Probabilistic Models, Solutions 4

1(a) $P(X_1 = 4) = (1/2)^4 = 0.0625$ and

$$\begin{aligned} P(X_2 = 4) &= \sum_{k=1}^{3} P(X_2 = 4 | X_1 = k) P(X_1 = k) \\ &= \sum_{k=1}^{3} \left(\frac{k}{4}\right)^4 \binom{4}{k} \left(\frac{1}{2}\right)^4 \\ &= 0.1035 \end{aligned}$$

Note that if $X_1 = 0$, there are no As in the first generation and that if $X_1 = 4$, A cannot *become* fixed in the second generation; it already is fixed.

(b) It is fixed in the second generation if it becomes fixed in the first or second generation. Since these are disjoint events, their probabilities can be added to give the answer $0.0625 + 0.1035 = 0.166$.

(c) $P(X_1 = 0) + P(X_1 = 4) = 0.125$ (twice the value in **(1)** since there are the same initial numbers of A and a).

(d) With $N_1 = 2$ we get the probability $2 \cdot (1/2)^2 = 1/2$ and with $N_1 = 8$ we get $2 \cdot (1/2)^8 = 0.0078$. When the population halves, the probability increases by a factor 4 and when population doubles, it decreases by a factor 16.

(e)

$$\left(\frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\right)\right)^{-1} = \frac{8}{3} \approx 3$$

$$\left(\frac{1}{2}\left(\frac{1}{4} + \frac{1}{8}\right)\right)^{-1} = \frac{16}{3} \approx 5$$

The averages are 3 and 6, respectively.

2(a) $X_k \sim \text{bin}\left(2N, \frac{1}{2N}\right)$

(b) In each draw, all individuals are equally likely to be chosen so the distribution is the same. Since the X_k must add up to $2N$ they can not be independent. If we for example know that $X_1 = 2N$, we know that the others must equal 0.

3(a) The variance in the bin(n, p)-distribution is $np(1-p)$ and here we get

$$\sigma^2 = 2N \cdot \frac{1}{2N}\left(1 - \frac{1}{2N}\right) = 1 - \frac{1}{2N} \to 1$$

as $N \to \infty$. We get $E[T] \approx 4N$ which agrees with our previous result.

(b) A large variance means that some individuals tend to have many children which also means that more individuals have no children. Thus, when individuals are sampled they are more likely to stem from the same parent if the variance is large.

(c) In this case all individuals always have exactly one offspring which means that no two individuals can ever have the same parent.