Probabilistic Models, HW5, due April 19

1. Let X be a random variable which takes on the values $0, 1$, and 2 with equal probabilities (uniform distribution on $\{0, 1, 2\}$.

(a) Find the pgf of X.
(b) Use the pgf in **(a)** to find the mean and variance of X.
(c) Let Y be independent of X and have the same distribution as X. Consider $X + Y$ which has range $\{0, 1, 2, 3, 4\}$. Find the pgf of $X + Y$.
(d) Use the pgf in **(c)** to decide whether $X + Y$ is uniform on its range.
(e) Use the pgf in **(c)** to find $P(X + Y = 0)$.

2. Let X have pgf G_X and let $Y = 1 + X$.

(a) Show that the pgf of Y, G_Y, is given by $G_Y(s) = sG_X(s)$ (you need to think of the constant 1 as a random variable).
(b) Use the relation between G_X and G_Y to show that $E[Y] = 1 + E[X]$.

3. Recall that if $X \sim$ bin (n, p), X can be written as the sum S_n of n independent indicators. Now let N have a Poisson distribution with mean n and consider the sum S_N (where the empty sum S_0 equals 0 as usual). Use pgf's to find the distribution of S_N. What is it called?

4. Consider a cell population where cells die with probability 1/3 and divide with probability 2/3.

(a) Find the extinction probability of a population started from a single ancestor.
(b) Find the extinction probability of a population started from two ancestors.
(c) If you can choose the number of ancestors and want the extinction probability to be at most 1%, how many ancestors should you have?

5. Find the extinction probabilities for the following branching processes.

(a) Cells divide with probability 4/5 and die with probability 1/5 (thus, X can be 0 or 2).
(b) Cells are c times as likely to divide as they are to die where $c > 1$.
(c) Cells are c times as likely to die as they are to divide where $c > 1$.
(d) X is equally likely to be 0 or 3.