Probabilistic Models, Solutions 5

1(a) $G(s) = p_0 + p_1 s + p_2 s^2 = (1 + s + s^2)/3$.

(b) $G'(s) = (1 + 2s)/3$ and $G''(s) = 2/3$ so that

$$E[X] = G'(1) = 1$$

and

$$\mathrm{Var}[X] = G''(1) + G'(1) - G'(1)^2 = \frac{2}{3}$$

(c) Since $G_{X+Y}(s) = G_X(s)G_Y(s)$ we get

$$G_{X+Y}(s) = (1 + s + s^2)^2/9 = (1 + 2s + 3s^2 + 2s^3 + s^4)/9$$

(d) The range is $\{0, 1, 2, 3, 4\}$ and a uniform distribution has pgf $(1 + s + s^2 + s^3 + s^4)/5$ which is not the pgf of $X + Y$. Hence, $X + Y$ is not uniform.

(e) $P(X + Y = 0) = G_{X+Y}(0) = \dfrac{1}{9}$

2(a) Recall that if C and X are independent random variables we have $G_{C+X}(s) = G_C(s)G_X(s)$. In particular, if C is constant, it is trivially independent of X (its distribution does not change if we observe the outcome of X). If $C = k$ with probability 1, it has pgf $G_C(s) = s^k$ and since we have $k = 1$, we get $G_{C+X}(s) = sG_X(s)$.

(b) Differentiate G_Y:

$$G'_Y(s) = G_X(s) + sG'_X(s)$$

Set $s = 1$ to get

$$E[Y] = G'_Y(1) = G_X(1) + G'_X(1) = 1 + E[X]$$

3. The sum S_N has pgf

$$G_{S_N}(s) = G_N(G_X(s))$$

where

$$G_X(s) = 1 - p + ps$$

and

$$G_N(s) = e^{n(s-1)}$$

and we get

$$G_{S_N}(s) = e^{n(G_X(s)-1)}$$

$$= e^{n(1-p+ps-1)} = e^{np(s-1)}$$

which is the pgf of a Poisson distribution with mean np, thus $S_N \sim \text{Poi}(np)$.

4(a) The pgf is $G(s) = \dfrac{1}{3} + \dfrac{2}{3}s^2$ and the equation $s = G(s)$ has solutions 1 and $1/2$ and thus $q = 1/2$.

(b) If there are two ancestors, the extinction probability is $(1/2)^2 = 1/4$ (since two independent branching processes must both go extinct).

(c) If there are k ancestors, the extinction probability is $(1/2)^k$ (since k independent branching processes must all go extinct) which will be ≤ 0.01 if $k \geq 7$.

5(a) Special case of (b) below with $c = 4$ thus, $P(E) = 1/4$.

(b) We have $p_2 = cp_0$ and $p_0 + p_2 = 1$ which gives $p_0 = 1/(1+c)$ and $p_2 = c/(1+c)$. The equation $s = G(s)$ becomes

$$s = \frac{1}{1+c} + s^2 \frac{c}{1+c}$$

which has solutions 1 and $1/c$. As $c > 1$ we have $P(E) = 1/c$ (the smallest solution in $[0,1]$).

(c) Here, $\mu = 2/(1+c) < 1$ so $P(E) = 1$.

(d) The equation $s = G(s)$ becomes

$$s = \frac{1}{2} + \frac{1}{2}s^3$$

which has solutions $1, (\sqrt{5}-1)/2$, and $-(\sqrt{5}-1)/2$. Note that $s = 1$ is always a solution to $s = G(s)$ which helps you factor the cubic. We get $P(E) = (\sqrt{5}-1)/2 \approx 0.61$ (and you may recognize the golden ratio).