Probability Models, Solutions 1

1(a) Since $\binom{15}{2} = 15 \cdot 14/2 = 105$ we get

$$S = 1 - (1-p)^{\binom{K}{2}} = 1 - 0.99^{1}05 = 0.65$$

and

$$E[I] = \binom{K}{2} p = 105 \cdot 0.01 = 1.05$$

(b) We must have

$$S = 1 - 0.99^{\binom{K}{2}} \geq 0.9$$

which gives

$$\binom{K}{2} \log 0.99 \leq \log 0.1$$

As $\log 0.99 < 0$, we get

$$\binom{K}{2} \geq \frac{\log 0.1}{\log 0.99} = 229.1$$

Trial and error gives that we must have $K \geq 22$ (or solve the quadratic).

2(a) Among the first 3 subsitutions there are 3 pairs that cannot cause any incompatibilities. The probability of this is $(1-p)^3$. The 4th substitution must then be incompatible with *at least* one of the 3 previous which has probability $1-(1-p)^3$. As incompatibilities between different gene pairs are assumed independent, we multilpy these two probabilities to get the answer.

2(b) Among the first $j-1$ substitutions there are $\binom{j-1}{2}$ pairs that cannot cause any incompatibilities. The probability of this is $(1-p)^{\binom{j-1}{2}}$. The jth must then be incompatible with *at least* one of the $j-1$ previous ones which has probability $1-(1-p)^{j-1}$. We get

$$P(K_S = j) = (1-p)^{\binom{j-1}{2}} \left(1 - (1-p)^{j-1}\right)$$