Probability Models, Solutions 1

1(a) Since \( \binom{15}{2} = 15 \cdot 14/2 = 105 \) we get
\[
S = 1 - (1 - p)^{\binom{K}{2}} = 1 - 0.99^{105} = 0.65
\]
and
\[
E[I] = \binom{K}{2} p = 105 \cdot 0.01 = 1.05
\]

(b) We must have
\[
S = 1 - 0.99^{\binom{K}{2}} \geq 0.9
\]
which gives
\[
\binom{K}{2} \log 0.99 \leq \log 0.1
\]
As \( \log 0.99 < 0 \), we get
\[
\binom{K}{2} \geq \frac{\log 0.1}{\log 0.99} = 229.1
\]
Trial and error gives that we must have \( K \geq 22 \) (or solve the quadratic).

2(a) Among the first 3 substitutions there are 3 pairs that cannot cause any incompatibilities. The probability of this is \( (1 - p)^3 \). The 4th substitution must then be incompatible with at least one of the 3 previous which has probability \( 1 - (1 - p)^3 \). As incompatibilities between different gene pairs are assumed independent, we multiply these two probabilities to get the answer.

2(b) Among the first \( j - 1 \) substitutions there are \( \binom{j-1}{2} \) pairs that cannot cause any incompatibilities. The probability of this is \( (1 - p)^{\binom{j-1}{2}} \). The \( j \)th must then be incompatible with at least one of the \( j - 1 \) previous ones which has probability \( 1 - (1 - p)^{j-1} \). We get
\[
P(K_S = j) = (1 - p)^{\binom{j-1}{2}} \left(1 - (1 - p)^{j-1}\right)
\]