Probability Models, Solutions 2

Turn-in problems:

1(a)

$$P(\text{speciation}) \approx 1 - (1-p)^{\alpha\binom{K}{2}} = 1 - 0.9^{0.0087 \cdot 45} \approx 0.04$$

(b)

$$P(\text{speciation}) = (1-p)^{\binom{K}{2}} = 1 - 0.0^{45} \approx 0.99$$

(c) We would need K large enough to make

$$1 - 0.9^{0.0087 \cdot \binom{K}{2}} \geq 0.99$$

that is

$$\binom{K}{2} \geq \frac{\ln 0.01}{\ln 0.9 \cdot 0.0087} \approx 5024$$

which gives $K \geq 101$. You can find this by trial and error, by solving the quadratic $K(K-1)/2 = 5024$ or by approximating $\binom{K}{2}$ by $K^2/2$. Your answer may vary slightly but not much depending on how you do it.

2. First note that $\alpha = 5/\binom{6}{2} = \frac{1}{3}$ and hence $E[X] = \frac{1}{3} \cdot \binom{3}{2} = 1$ for all networks.

(a,b) Network I:

$P(X = 0) = 0.5$
$P(X = 1) = 0$
$P(X = 2) = 0.5$
$P(X = 3) = 0$ $\qquad E[X] = 0 \cdot 0.5 + 1 \cdot 0 + 2 \cdot 0.5 + 3 \cdot 0 = 1$

Network II:

$P(X = 0) = 0.3$
$P(X = 1) = 0.5$
$P(X = 2) = 0.1$

$P(X = 3) = 0.1 \qquad E[X] = 0 \cdot 0.3 + 1 \cdot 0.5 + 2 \cdot 0.1 + 3 \cdot 0.1 = 1$

Network III:

$P(X = 0) = 0.2$
$P(X = 1) = 0.6$
$P(X = 2) = 0.2$
$P(X = 3) = 0 \qquad E[X] = 0 \cdot 0.2 + 1 \cdot 0.6 + 2 \cdot 0.2 + 3 \cdot 0.1 = 1$

(c) The exact formula is

$$P(\text{speciation}) = 1 - \sum_{j=0}^{3} (1-p)^j P(X = j)$$

where $p = 0.1$ and the probabilities $P(X = j)$ were computed in **(a)** above. The three speciation probabilities are $0.095, 0.0961$, and 0.098, for networks I, II, and III, respectively. The approximation formula is

$$P(\text{speciation}) \approx 1 - (1-p)^{\alpha\binom{K}{2}}$$

where $K = 3$ and $\alpha = \dfrac{1}{3}$ from above. Hence $P(\text{speciation}) \approx 0.1$ for all three networks.

3(a) First note that we can avoid edges in our sample only if our chosen nodes are on different edges. Thus, if $K > N_E$, we cannot avoid getting at least one edge in our sample and hence $P(X = 0) = 0$.

(b) With $N = 6$ we get $N_E = 3$. Our 2 nodes must be chosen on different edges and as there are 3 edges, we have $\binom{3}{2} = 3$ ways of choosing the 2 edges our nodes can be on. As each edge has 2 nodes, we have $2 \cdot 2 = 4$ choices of nodes for each edge pair. Thus, there is a total of $3 \cdot 4 = 12$ ways of choosing 2 nodes so that there are no edges in our sample. As the total number of ways to choose 2 out of 6 nodes is $\binom{6}{2} = 15$, the probability is $12/15 = 4/5$.

(c) Arguing as in **(b)**, we have $\binom{N_E}{K} = \binom{N/2}{K}$ ways to choose the K edges are nodes are on. For each such choice, there are 2^K ways of choosing nodes on these edges. Hence we get

$$P(X=0) = \frac{\binom{N/2}{K} \cdot 2^K}{\binom{N}{K}}$$