Probability Models, Solutions 3

Turn-in problems:

1.

(a) Use the formulas $N_S = \sum_{j=2}^{N-1} n_j \binom{j}{2}$ where n_j is the number of nodes of degree j, and $N_D = \binom{N_E}{2} - N_S = \binom{5}{2} - N_S = 10 - N_S$.

Network I: Here we only have $n_5 = 1$ and hence $N_S = \binom{5}{2} = 10$. Next, $N_D = \binom{N_E}{2} - N_S = \binom{5}{2} - 10 = 0$ (obviously there are no disjoint edge pairs in this network).

Network II: We have $n_3 = 2$ and $n_2 = 2$ which gives $N_S = 2 \cdot \binom{3}{2} + 2 \cdot \binom{2}{2} = 8$ and hence $N_D = 2/$

Network III: We have $n_2 = 4$ which gives $N_S = 4 \cdot \binom{2}{2} = 4$ and hence $N_D = 6$.

(b) Since we choose $K = 3$ nodes, we cannot get 2 disjoint edges which requires 4 nodes. Formally, $\binom{3}{4} = 0$.

(c) Since $P_4 = 0$ for our networks the formula is

$$\mathrm{Var}[X] = N_E P_2(1 - P_2) + 2(N_S(P_3 - P_2^2) - N_D P_2^2)$$

where all networks have $N_E = 5$ edges and

$$P_2 = \frac{\binom{K}{2}}{\binom{N}{2}} = \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{3}{15} = 0.2$$

$$P_3 = \frac{\binom{K}{3}}{\binom{N}{3}} = \frac{\binom{3}{3}}{\binom{6}{3}} = \frac{1}{20} = 0.05$$

Network I: Here $N_S = 10$ and $N_D = 0$ which gives

$$\mathrm{Var}[X] = 5 \cdot 0.2 \cdot (1 - 0.2) + 2 \cdot 10 \cdot (0.05 - 0.2^2) = 1$$

Network II: Here $N_S = 8$ and $N_D = 2$ which gives

$$\text{Var}[X] = 5 \cdot 0.2 \cdot (1 - 0.2) + 2 \cdot (8 \cdot (0.05 - 0.2^2) - 2 \cdot 0.2^2) = 0.8$$

Network III: Here $N_S = 4$ and $N_D = 6$ which gives

$$\text{Var}[X] = 5 \cdot 0.2 \cdot (1 - 0.2) + 2 \cdot (4 \cdot (0.05 - 0.2^2) - 6 \cdot 0.2^2) = 0.4$$

(d) Use the regular variance formula

$$\text{Var}[X] = E[X^2] - (E[X])^2$$

(e) The formula for the variance is

$$\text{Var}[S] \approx (\ln(1-p))^2 (1-p)^{2\mu} \sigma^2$$

where $\mu = E[X] = 1$ in all networks and $\sigma^2 = \text{Var}[X]$. Also recall that $p = 0.1$ in all networks.

Network I: Here $\sigma^2 = 1$ which gives

$$\text{Var}[S] \approx (\ln(0.9))^2 \cdot 0.9^2 \cdot 1 = 0.009$$

Network II: Here $\sigma^2 = 0.8$ which gives

$$\text{Var}[S] \approx (\ln(0.9))^2 \cdot 0.9^2 \cdot 0.8 = 0.0072$$

Network III: Here $\sigma^2 = 0.4$ which gives

$$\text{Var}[S] \approx (\ln(0.9))^2 \cdot 0.9^2 \cdot 0.4 = 0.0036$$

2. We have $N = 15$ and $N_E = 19$. Recall the formula

$$N_S = \sum_{j=2}^{N-1} n_j \binom{j}{2}$$

where n_j is the number of nodes of degree j. We get $N_S = 37$ (which can of course also be obtained by direct counting but it is easy to make mistakes). Finally, $N_D = \binom{N_E}{2} - N_S = 171 - 37 = 134$.