Probability Models, Solutions 4

1(a) $P(X_1 = 4) = (3/4)^4 = 0.316$ and

$$P(X_2 = 4) = \sum_{k=1}^{3} P(X_2 = 4|X_1 = k)P(X_1 = k)$$

$$= \sum_{k=1}^{3} \left(\frac{k}{4}\right)^4 \binom{4}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{4-k}$$

$$= 0.147$$

Note that if $X_1 = 0$, there are no As in the first generation and that if $X_1 = 4$, A cannot *become* fixed in the second generation; it already is fixed in the first.

(b) It is fixed in the second generation if it becomes fixed in the first or second generation. Since these are disjoint events, their probabilities can be added to give the answer $0.316 + 0.147 = 0.46$.

(c) $P(X_1 = 4) + P(X_1 = 0) = (3/4)^4 + (1/4)^4 = 0.32$

(d) Let $2N_1$ denote the size of the first generation so that $X_1 \sim \text{bin}(2N_1, 3/4)$. With $2N_1 = 2$ we get the probability

$$P(X_1 = 0) + P(X_1 = 4) = (1/4)^2 + (3/4)^2 = 0.625$$

and with $2N_1 = 8$ we get

$$P(X_1 = 0) + P(X_1 = 4) = (1/4)^8 + (3/4)^8 = 0.1$$

2(a) Since each individual has a $1/2N$ probability to be chosen in any trial and there are $2N$ trials, $X_k \sim \text{bin}\left(2N, \frac{1}{2N}\right)$

(b) Since the X_k must add up to $2N$ they can not be independent. If we for example know that $X_1 = 2N$, we know that the others must equal 0.

(c) The smallest possible value is 0 if every individual has 0 offspring, and the largest possible value is $2N \cdot 2N = 4N^2$ if every individual has $2N$ offspring. All values between 0 and $4N^2$ are possible. In fact, since we are doing $2N$

binomial trials for each of the the $2N$ individuals, we are doing a total of $4N^2$ trials and the distribution is $\text{bin}\left(4N^2, \frac{1}{2N}\right)$.