Probability Models, Solutions 5

1(a) $P(X_1 = 6) = (4/6)^6 = 0.09$
(b) $P(X_1 = 0) = (2/6)^6 = 0.0014$

Average: $(6 + 12)/2 = 9$

Effective population size:

$$N_e = \frac{1}{\frac{1}{2}\left(\frac{1}{6} + \frac{1}{12}\right)} = 8$$

2. The harmonic mean equals

$$\frac{1}{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{n}\right)}$$

Since $\frac{1}{n} > 0$, the denominator is greater than $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and hence the expression is less than 4 for any n.

3(a) The variance in the bin(n, p)-distribution is $np(1 - p)$ and here we get

$$\sigma^2 = 2N \cdot \frac{1}{2N}\left(1 - \frac{1}{2N}\right) = 1 - \frac{1}{2N} \to 1$$

(b) The number of offspring X of an individual is either 0 or 2, with equal probabilities. Hence, the mean is $E[X] = 0 \cdot (1/2) + 2 \cdot (1/2) = 1$ and by the variance formula

$$\text{Var}[X] = E[X^2] - (E[X])^2 = E[X^2] - 1$$

where

$$E[X^2] = 0^2\frac{1}{2} + 2^2\frac{1}{2} = 2$$

so that $\text{Var}[X] = 1$. We get $E[T] \approx 4N$.

(c) A large variance means that some individuals tend to have many children which also means that more individuals have no children. Thus, when individuals are sampled they are more likely to stem from the same parent if the variance is large.

(c) In this case all individuals always have exactly one offspring which means that no two individuals can ever have the same parent.