Probability Models, solutions to practice problems

1(a) \( G(s) = p_0 + p_1 s + p_2 s^2 = \frac{(1 + s + s^2)}{3}. \)

(b) \( G'(s) = \frac{(1 + 2s)}{3} \) and hence \( E[X] = G'(1) = 1. \)

(c) Since \( G_{X+Y}(s) = G_X(s)G_Y(s) \) we get

\[
G_{X+Y}(s) = \frac{(1 + s + s^2)^2}{9} = \frac{(1 + 2s + 3s^2 + 2s^3 + s^4)}{9}
\]

(d) \( P(X + Y = 0) = G_{X+Y}(0) = \frac{1}{9} \)

2(a) The pgf is \( G(s) = \frac{1}{3} + \frac{2}{3} s^2 \) and the equation \( s = G(s) \) has solutions 1 and \( 1/2 \) and thus \( q = 1/2. \)

(b) If there are two ancestors, the extinction probability is \((1/2)^2 = 1/4\) (since two independent branching processes must both go extinct).

(c) If there are \( k \) ancestors, the extinction probability is \((1/2)^k\) (since \( k \) independent branching processes must all go extinct) which will be \( \leq 0.01 \) if \( k \geq 7. \)

3(a) The mean number of offspring is \( 0 \cdot 0.8 + 2 \cdot 0.2 = 0.4 \) which is less than 1 and \( P(E) = 1. \)

(b) The mean number of offspring is \( 0 \cdot (1 - p) + 1 \cdot p = p < 1 \) so \( P(E) = 1. \)

(c) The mean number of offspring is \( G'(1) \) and as \( G'(s) = e^{s-1} \), we have \( G'(1) = 1 \) and hence \( P(E) = 1. \)

(d) This pgf tells us that an individual has 1 offspring with probability 0.2 or 4 offspring with probability 0.8. As all individuals have offspring, the population cannot go extinct so \( P(E) = 0. \) Alternatively, solve the equation \( s = 0.2s + 0.8s^4 \) to get the solutions \( s = 0 \) and \( s = 1 \), the smaller of which is the extinction probability.