Probability Models, solutions to practice problems

1(a) $G(s) = p_0 + p_1 s + p_2 s^2 = (1 + s + s^2)/3$.

(b) $G'(s) = (1 + 2s)/3$ and hence $E[X] = G'(1) = 1$.

(c) Since $G_{X+Y}(s) = G_X(s)G_Y(s)$ we get

$$G_{X+Y}(s) = (1 + s + s^2)^2/9 = (1 + 2s + 3s^2 + 2s^3 + s^4)/9$$

(d) $P(X + Y = 0) = G_{X+Y}(0) = \frac{1}{9}$

2(a) The pgf is $G(s) = \frac{1}{3} + \frac{2}{3}s^2$ and the equation $s = G(s)$ has solutions 1 and 1/2 and thus $q = 1/2$.

(b) If there are two ancestors, the extinction probability is $(1/2)^2 = 1/4$ (since two independent branching processes must both go extinct).

(c) If there are k ancestors, the extinction probability is $(1/2)^k$ (since k independent branching processes must all go extinct) which will be ≤ 0.01 if $k \geq 7$.

3(a) The mean number of offspring is $0 \cdot 0.8 + 2 \cdot 0.2 = 0.4$ which is less than 1 and $P(E) = 1$.

(b) The mean number of offspring is $0 \cdot (1 - p) + 1 \cdot p = p < 1$ so $P(E) = 1$.

(c) The mean number of offspring is $G'(1)$ and as $G'(s) = e^{s-1}$, we have $G'(1) = 1$ and hence $P(E) = 1$.

(d) This pgf tells us that an individual has 1 offspring with probability 0.2 or 4 offspring with probability 0.8. As all individuals have offspring, the population cannot go extinct so $P(E) = 0$. Alternatively, solve the equation $s = 0.2s + 0.8s^4$ to get the solutions $s = 0$ and $s = 1$, the smaller of which is the extinction probability.