MATH 332, Prob & Stat, Solutions to Test 1

1. (a), (c), (f) are always true.

For (a) and (c), use Venn diagrams; (f) follows from the definition of independence of 3 events; (b) is only true for independent events; a counterexample to (d) is to let $A = \{1\}$ and $B = \{2\}$ in a die roll which has $P(A|B) = 0$ and $P(A|B^c) = 1/5$; a counterexample to (e) is given in Problem 50, p.71 in the book.

2(a) $P(A \cup B) = P(A) + P(B) - P(A)P(B) = 5/8$, $P(A \cap B) = P(A)P(B) = 1/8$
(b) $P(A \cup B) = P(A) + P(B) = 3/4$, $P(A \cap B) = 0$
(c) Draw a Venn diagram to argue that $A \cap B^c = \emptyset$ implies that $A \subseteq B$. Thus, $P(A \cup B) = P(B) = 1/2$ and $P(A \cap B) = P(A) = 1/4$.

3. To avoid aces, you have 12 cards in each suit to choose from. The hearts can be chosen in $\binom{12}{5}$ ways, the diamonds in $\binom{12}{3}$ ways and the remaining 5 cards in $\binom{24}{5}$ ways. As the total number of bridge hands is $\binom{52}{13}$, the answer is

$$\frac{\binom{12}{5}\binom{12}{3}\binom{24}{5}}{\binom{52}{13}}$$

4(a) Let D denote having the disease; let $+$ and $-$ denote testing positive and negative, respectively. We are given the probabilities $P(D) = 1/10000 = 0.0001$, $P(+|D) = 1$, and $P(-|D^c) = 0.99$. By LTP,

$$P(+) = P(+|D)P(D) + P(+|D^c)P(D^c) = 1 \cdot 0.0001 + 0.01 \cdot 0.999 \approx 0.01$$

(b) Bayes' formula gives

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)} = \frac{1 \cdot 0.0001}{1 \cdot 0.0001 + 0.01 \cdot 0.999} \approx 0.01$$

5. Condition on the next two changes. With probability $2/3 \cdot 2/3 = 4/9$, both changes are both completions of jobs and the busy period is over without any lost jobs. With probability $1/3 \cdot 1/3 = 1/9$, both changes are arrivals of new jobs and the second of these jobs is lost. Finally, with probability $1-(4/9+1/9)=4/9$, there is one completion and one arrival, in any order, and the system starts over at two jobs. Formally, let A be the event that no jobs are lost before the current busy period ends and let

$B_1 = \{\text{next two changes are completions}\}$
$B_2 = \{\text{next two changes are arrivals}\}$
$B_3 = \{\text{next two changes are one of each}\}$

Now let $q = P(A)$ and note that $P(A|B_1) = 1$ and $P(A|B_2) = 0$. In the third case, the system starts over with 2 jobs and thus $P(A|B_3) = q$. LTP gives

$$q = P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= 1 \cdot 4/9 + 0 \cdot 1/9 + q \cdot 4/9$$

and we get the equation $q = 4/9 + 4q/9$ which has solution $q = 4/5$.

6. For any x, $F(x) = P(X \le x)$. Moreover, the probability $P(X = x)$ is the size of the jump in F at the point x. Thus

(a) $P(X = 1) = 1/4$ **(b)** $P(X = 2) = 1/2$ **(c)** $P(X = 2.5) = 0$ **(d)** $P(X \le 2.5) = 3/4$

Note that the cdf F tells us that X can take on the values 1, 2, and 3 with probabilities $1/4, 1/2$, and $1/4$, respectively.