1. Consider a branching process where an individual can have 0, 1, or 2 offspring with equal probabilities. Find $E[Z_n]$, $P(Z_2 = 0)$, and $\lim_{n \to \infty} P(Z_n = 0)$.

2. Find the extinction probabilities for the following branching processes.
   (a) $P(X = k) = \frac{90}{\pi^4} \frac{1}{k^4}$, $k = 1, 2, ...$
   (b) The offspring distribution is the so-called truncated geometric distribution which has
      
      \[
      P(X = 0) = p_0
      
      P(X = k) = cp^k, \quad k = 1, 2, ...
      \]
      
      where $p_0, p$, and $c$ must satisfy
      \[
      p_0 + c \frac{p}{1 - p} = 1
      \]
      
      and it can be shown that the mean is
      \[
      E[X] = \frac{cp}{(1 - p)^2}
      \]
      
      Let $p_0 = 2/3$ and $p = 1/2$.
   (c) The offspring distribution has pgf $G(s) = e^{-0.5+0.5s}$
   (d) An individual is equally as likely to have 0 or 3 offspring. (Do you recognize the solution?)

3(a) In HW5 you established the relation $Q_n(s) = sG(Q_{n-1}(s))$ where $Q_n$ is the pgf of $Y_n$ = the total number of individuals in the first $n$ generations including the ancestor and $G$ is the pgf of the offspring distribution. Since
\( Y_n \) is nonnegative and increasing it converges to some random variable \( Y \) as 
\( n \to \infty \); hence \( Y \) is the total number of individuals ever born. It can be 
shown that \( Y \) has pgf \( Q \) given by \( Q(s) = \lim_{n \to \infty} Q_n(s) \). If \( \mu \leq 1 \), we know 
that the population goes extinct and thus \( Y < \infty \) in this case. If \( \mu > 1 \), 
there is a positive probability that \( Y = \infty \) and pgf methods cannot be used. 
Suppose that \( \mu \leq 1 \) and show that

\[
Q'(s) = \frac{G(Q(s))}{1 - sG'(Q(s))}
\]

and use this relation to find \( E[Y] \). What is the difference between the sub-
critical and critical cases?

(b) Consider a cell population where a cell is equally as likely to die as it is 
to split into 2 offspring. Find \( Q(s) \). \textit{Hint:} A quadratic equation will show 
up. In order to decide between the two solutions, interpret \( Q(0) \) as the limit 
of \( Q(s) \) as \( s \to 0 \).

4. Consider a queueing system where arrivals occur according to a Poisson 
process with rate 1 and service times are independent \( \sim \exp(1) \). There are 3 
servers and no waiting room (customers who arrive when all servers are busy 
are lost).

(a) Sketch the rate diagram for the continuous-time chain and the transition 
diagram for the jump chain.

(b) State the generator for the continuous-time chain and the transition ma-
trix for the jump chain.

(c) Find the stationary distribution \( \pi \) for the continuous-time chain and the 
stationary distribution \( \nu \) for the jump chain. Verify that \( \nu_k \) is proportional 
to \( \lambda(k) \cdot \pi_k \) for \( k = 0, 1, 2, 3 \).

(d) In a given 2-hour period, how many minutes can you expect the system 
to be empty?
(e) Now add room for one customer to wait in line and suppose that an arriving customer who finds all servers busy stays to wait in line with probability $p$, and leaves with probability $1 - p$. Sketch the rate diagram for the continuous-time chain and the transition graph for the jump chain.

5. Consider a bacterial population where individuals have lifetimes that are $\exp(\alpha)$. There is immigration into the population such that groups of immigrants arrive according to a Poisson process with rate $\lambda$. A group is of size $k$ with probability $p_k$ for $k \geq 1$. Individual bacteria reproduce by dividing where the probability that a bacterium in a population of size $k$ divides rather than dies at the end of its lifetime equals $q_k$ for $k \geq 1$. As long as $k \leq 10^4$ there is no death, that is, $q_k = 1$ for $k \leq 10^4$. Occasionally, there is a disaster that kills the entire bacterial population. Suppose that disasters occur according to a Poisson process with rate $\mu = 0.1\lambda$. What is the probability that an arriving immigrant finds the population extinct?