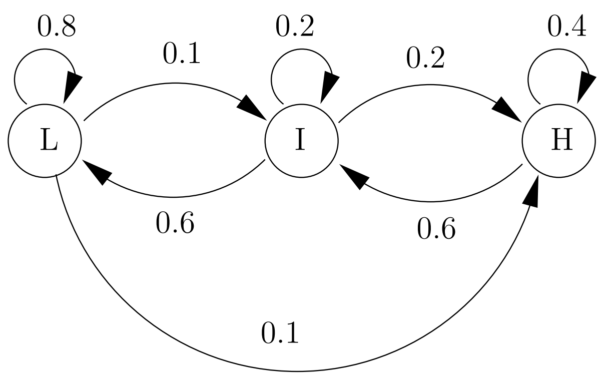
Stochastic Processes, Solutions 1

Book 3. $p_{dd} = 0.3, p_{dn} = 0.7, p_{nd} = 0.01, p_{nn} = 0.99$.

Book 4(a): The transition graph is



and the transition matrix is (in the order L, I, H)

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$

Book 6. Yes. For a simple example, let $S = \{0, 1\}$ and let $p_{01} = 1, p_{11} = 1$.

Book 7. Let $p_{i,i+1} = 1$ for $i = 1, 2, ..., n-1$ and $p_{nn} = 1$; only n is then recurrent.

2(a) All states are recurrent, one equivalence class $\{1, 2, 3\}$, the chain is irreducible.
(b) State 3 is recurrent, states 1 and 2 are transient, three classes $\{1\}, \{2\}, \{3\}$, the chain is not irreducible (and is then called reducible).
(c) States 1 and 3 are recurrent, state 2 is transient, two classes $\{1, 3\}$ and $\{2\}$, not irreducible.
(d) States 1 and 3 are recurrent, states 2, 4, 5, 6 are transient, four classes $\{1, 3\}$, $\{2\}$, $\{4\}$, $\{5, 6\}$, not irreducible.

Turn-in problems

1. State space: $\{r, s\}$, possible outcomes for 2 consecutive days: $\{rr, rs, sr, ss\}$ with probabilities $0.1, 0.1, 0.1$, and 0.7, respectively. By definition of conditional probability, $P(B|A) = P(A \cap B)/P(A)$ which gives

$$p_{ss} = P(X_1 = s|X_0 = s) = \frac{P(X_0 = s, X_1 = s)}{P(X_0 = s, X_1 = s) + P(X_0 = s, X_1 = r)}$$

$$= \frac{0.7}{0.7 + 0.1} = \frac{7}{8}$$

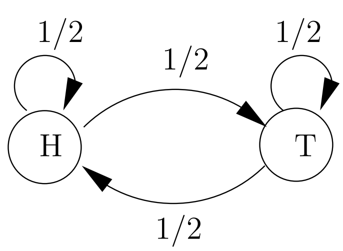
which gives $p_{sr} = 1/8$. Similarly, $p_{rr} = 1/2, p_{rs} = 1/2$. Make sure you understand the difference between *joint* probabilities of the type $P(A \cap B)$ and *conditional* probabilities of the type $P(B|A)$.

2. False. For a trivial counterexample, consider the Markov chain on $\{0, 1\}$ that has $p_{00} = p_{11} = 1$ which has two recurrent states and two equivalence classes $\{0\}$ and $\{1\}$. Another counterexample is the gene example we did in class.

3(a) State space $S = \{H, T\}$. As successive flips are independent, we get $P(X_{n+1} = j|X_0 = i_0, ..., X_n = i) = P(X_{n+1} = j) = 1/2$ and $P(X_{n+1} = j|X_n = i) = 1/2$ as well for all states in S. The transition matrix is

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

and the transition graph



(b) State space $S = \{HH, HT, TH, TT\}$. Transition matrix (in that order):

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(c) There are only two possible paths from HH to TT in 3 steps: HH, HT, TT and HT, TT, TT which gives $P(X_3 = TT|X_0 = HH) = (1/2)^3 + (1/2)^3 = 1/4$. Alternatively, compute P^3 which has 1/4 at every entry. A third way of reasoning is that flips 3 and 4 are independent of flips 0 and 1, and the probability of getting two consecutive tails is 1/4.

4. We have to show that $P_1(\tau_1 < \infty) = 1$. We have $\tau_1 = 3 + X$ where X is the number of jumps from 2 to itself (where $X \geq 0$ and $\tau_1 \geq 3$) and get

$$P_1(\tau_1 = k) = P(X = k - 3) = (1 - p)p^{k-3}, \quad k \geq 3$$

which gives

$$P_1(\tau_1 < \infty) = \sum_{k=3}^{\infty} P(\tau_1 = k) = (1 - p) \sum_{k=3}^{\infty} p^{k-3} = \frac{1 - p}{1 - p} = 1$$

and state 1 is recurrent. If we start in state 2, the chain will eventually succeed to leave it (think geometric trials) and once it does, it returns after 2 steps. A similar argument holds for state 0 so all states are recurrent. Of course, by now you know that all states that communicate with a recurrent state are recurrent.