Stochastic Processes, Spring 2017, Test 1

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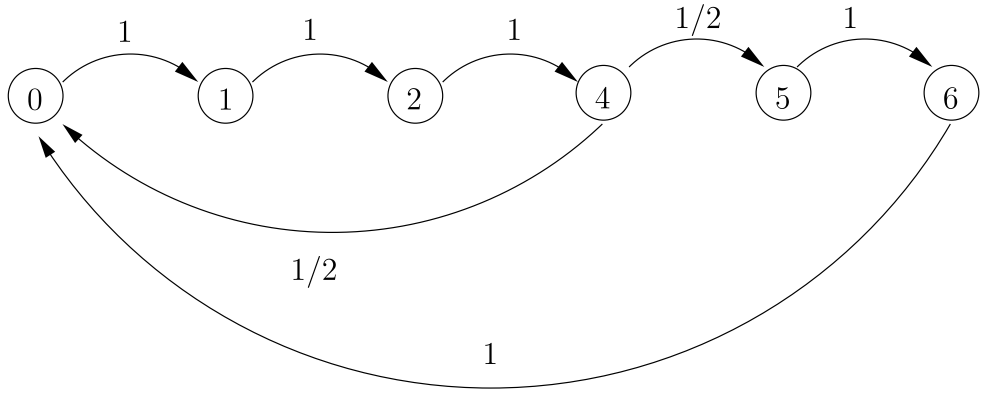
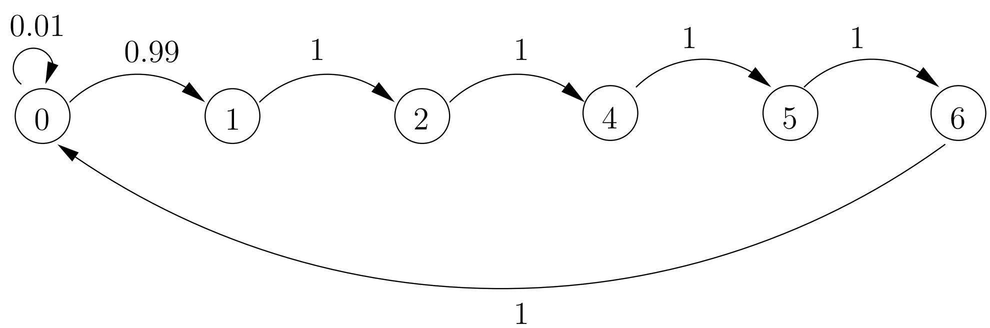
1. Find the periods of the two Markov chains below.





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2. Consider the "rain example" we have looked at in class. What is the probability to get 3 consecutive rainy days sometime in the distant future?

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3. Consider an irreducible and positive recurrent Markov chain on state space S where the initial state is chosen according to the stationary distribution π. What is the expected return time to the initial state if **(a)** S is finite with n states, **(b)** S is infinite. If you cannot solve it in general, you may compute it in some special cases for partial credit.

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4. Let $\{X_n, n = 0, 1, ...\}$ be a Markov chain with transition matrix P and define $Y_n = X_{3n}$.
(a) The sequence $\{Y_n, n = 0, 1, ...\}$ is also a Markov chain. What is its transition matrix?
(b) If $\{X_n\}$ is irreducible, must $\{Y_n\}$ be irreducible? Give proof or counterexample.
(c) If $\{Y_n\}$ is irreducible, must $\{X_n\}$ be irreducible? Give proof or counterexample.

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5. Consider a Markov chain on the state space $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

(a) Classify the states with respect to transience/null recurrence/positive recurrence and period. No formal proofs needed.
(b) Is the chain irreducible. Why/why not?
(c) Find the stationary distribution. Is it unique?
(d) Does $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$ for all i, j?

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6. Birds arrive one by one to rest on a branch of a tree where there is room for a maximum of 4 birds. As long as the birds feel safe they do not leave, but there are occasional incidents that scare them and then all of them leave at once. At any given time, suppose that a new arrival is 3 times as likely as a scary incident. If we observe the number of birds on the branch immediately after each change, we get a Markov chain with state space $S = \{0, 1, 2, 3, 4\}$.

(a) Give the transition graph and the transition matrix.
(b) Find the stationary distribution. Is it the limit distribution?
(c) What proportion of arriving birds turn away because the branch is full?
(d) If the branch is currently empty and an event (scary incident or arrival) occurs on average every 30 seconds, how much later can we expect the branch be empty again?
(e) If the branch is currently full, how many times can we expect it to be empty before it becomes full again?

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7(a) In the proof of Theorem 8.1, we showed that $|p_{ij}^{(n)} - \pi_j| \to 0$ as $n \to \infty$ for a fixed initial state i. Instead assume that we choose the initial state according to some probability distribution ν on S and wish to prove that $|p_{\nu j}^{(n)} - \pi_j| \to 0$ as $n \to \infty$. Adjust the coupling proof so that it works also in this case (not much adjustment is needed.)

(b) Use an example from your every-day life to explain the main ideas of coupling to somebody who does not know any mathematics. Truthfulness not required. Bonus points for entertainment value.

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