Stochastic Processes, Solutions to Test 1

1. First chain: $\{n : p_{00}^{(n)} > 0\} = \{4, 6, 8, ...\}$ which gives period 2.

Second chain: $\{n : p_{00}^{(n)} > 0\} = \{1, 2, 3, ...\}$ which gives period 1.

2. By the Markov property

$$P(X_n = r, X_{n+1} = r, X_{n+2} = r) =$$

$$P(X_n = r)P(X_{n+1} = r|X_n = r)P(X_{n+2} = r|X_{n+1} = r)$$

$$= P(X_n = r) \cdot p_{rr} \cdot p_{rr}$$

We have $p_{rr} = 0.8$ and for large n, $P(X_n = r) \approx \pi_r$ which gives the desired probability $2/3 \cdot 0.8^2 \approx 0.43$.

3. Let τ denote the return time to the initial state. Then

$$E_\pi[\tau] = \sum_{i \in S} E_i[\tau_i]\pi_i$$

because $E_i[\tau_i]\pi_i = 1$ for all i. Thus we get **(a)** $E_\pi[\tau] = n$ and **(b)** $E_\pi[\tau] = \infty$.

4(a) Since

$$P(Y_{n+1} = j|Y_n = i) = P(X_{3n+3} = j|X_{2n} = i) = P(X_3 = j|X_0 = i) = p_{ij}^{(3)}$$

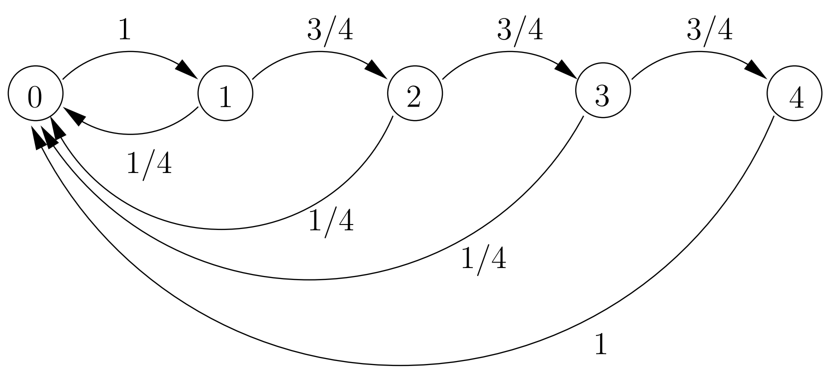
the transition matrix of $\{Y_n\}$ is P^3.

(b) No. Let $S = \{0, 1, 2\}$ and let $p_{01} = p_{12} = p_{20} = 1$. Then $\{X_n\}$ is irreducible but $\{Y_n\}$ is not because in the Y-chain each state can only reach itself.

(c) Yes. Denote the transition probabilities of $\{Y_n\}$ by q_{ij} so that $q_{ij} = p_{ij}^{(3)}$. If $\{Y_n\}$ is irreducible, for each pair of states i and j, there exist m and n such that $q_{ij}^{(m)} > 0$ and $q_{ji}^{(n)} > 0$ which means that $p_{ij}^{(3m)} > 0$ and $p_{ji}^{(3n)} > 0$ and $\{X_n\}$ is irreducible.

5(a) States 1 and 2 are positive recurrent; states 3 and 4 are transient. All states have period 1 (because they have $p_{ii} > 0$).
(b) No, because states 1 and 2 cannot reach states 3 and 4.
(c) Solving $\pi P = \pi$ under the condition $\sum_k \pi_k = 1$ gives the unique solution $\pi = (1/2, 1/2, 0, 0)$.
(d) Yes. Intuitively, the chain will eventually leave states 3 and 4 and spend all its time on the set $\{0, 1\}$. Formally, note that the 4 entries in the southeast corner of P^n go to 0 as $n \to \infty$ and thus the 4 entries in the southwest corner go to 1/2.

6. The transition graph is



and the equation $\pi P = \pi$ becomes

$$(\pi_0 ... \pi_4) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 1/4 & 0 & 0 & 3/4 & 0 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = (\pi_0 ... \pi_4)$$

which gives (skipping the first equation)

$$\pi_1 = \pi_0$$

$$\pi_2 = \frac{3}{4}\pi_1 = \frac{3}{4}\pi_0$$

$$\pi_3 = \frac{3}{4}\pi_2 = \frac{9}{16}\pi_0$$

$$\pi_4 = \frac{3}{4}\pi_3 = \frac{27}{64}\pi_0$$

which gives a stationary distribution of the form $\pi = \pi_0(1, 1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64})$ which gives

$$\pi_0 \left(1 + 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64}\right) = \pi_0 \frac{239}{64} = 1$$

which gives $\pi_0 = \dfrac{64}{239}$ and the stationary distribution

$$\pi = \left(\frac{64}{239}, \frac{64}{239}, \frac{48}{239}, \frac{36}{239}, \frac{27}{239}\right)$$

Since all states communicate, the chain is irreducible and since $p_{00}^{(2)} > 0$ and $p_{00}^{(3)} > 0$, the chain is aperiodic. Hence, π is the limit distribution.

(c) The proportion is $\pi_4 \approx 0.11$.

(d) The expected return time (in number of steps) is $1/\pi_0 = 239/64 \approx 3.73$ and with 30 seconds between events we get $3.73 \cdot 30 \approx 112$ seconds.

(e) $E_4[N_0] = \pi_0/\pi_4 = 64/27 \approx 2.4$.

7(a) The coupling inequality is now

$$|p_{\nu,j}^{(n)} - \pi_j| \leq P_{(\nu,\pi)}(T > n)$$

In the proof for a fixed initial state $X_0 = i$, it was

$$|p_{i,j}^{(n)} - \pi_j| \leq P_{(i,\pi)}(T > n)$$

where $P_{(i,\pi)}(T > n) \to 0$ as $n \to \infty$ because we had shown that

$$P_{(i,\pi)}(T < \infty) = 1$$

This in turn followed from positive recurrence of the chain $Z_n = (X_n, Y_n)$ under the probability distribution $P_{(i,\pi)}$. However, Z_n is not necessarily positive recurrent under $P_{(\nu,\pi)}$ as Problem 1(c) above shows. However, we do not actually need *positive* recurrence, only recurrence; that is, $P_{(\nu,\pi)}(T < \infty) = 1$ and this we have because

$$P_{(\nu,\pi)}(T < \infty) = \sum_{i \in S} P_{(i,\pi)}(T < \infty)\nu_i = \sum_{i \in S} \nu_i = 1$$

and we have

$$|p_{\nu,j}^{(n)} - \pi_j| \leq P_{(\nu,\pi)}(T > n) \to 0$$

as $n \to \infty$.

(b) Whatever anecdote you use, we want to show that a process has some "good property" (in our case, converging to the stationary distribution) by using another independent process. There are two main points of coupling: that the two processes meet with certainty, and that the other process has the good property in question.