Stochastic Processes, Solutions to Test 2

1(a) Recall that $\gamma_{ij} = \lambda(i)p_{ij}$ for $i \neq j$, and $\gamma_{ii} = -\lambda(i)$ to get the generator

$$G = \begin{pmatrix} -9 & 3 & 6 \\ 2 & -6 & 4 \\ 3 & 0 & -3 \end{pmatrix}$$

(b) Solving $\nu P = \nu$ gives

$$\nu = \left(\frac{9}{20}, \frac{3}{20}, \frac{8}{20}\right)$$

and solving $\pi G = 0$ gives

$$\pi = \left(\frac{6}{25}, \frac{3}{25}, \frac{16}{25}\right)$$

(d) The long-term proportion the chain spends in state 0 is $\pi_0 = 6/25$ or about 24% of the time.

2. The birth rates are $\lambda_k \equiv \lambda$ for $k \geq 0$ and the death rates are $\mu_k = k\mu$ (k servers busy) for $k \geq 1$. We get

$$\frac{\lambda_0 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \ldots \cdot \mu_n} = \frac{\rho^n}{n!}, \ n \geq 1$$

which gives $\pi_0 = e^{-\rho}$ and generally,

$$\pi_n = e^{-\rho} \cdot \frac{\rho^n}{n!}, \quad n = 0, 1, 2, \ldots$$

which is a Poisson distribution with mean ρ.

4. The birth rates are $\lambda_k = \dfrac{\lambda}{k+1}$ for $k \geq 0$ and the death rates are $\mu_k \equiv \mu$ for $k \geq 1$. We get

$$\frac{\lambda_0 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \ldots \cdot \mu_n} = \frac{\rho^n}{n!}, \ n \geq 1$$

which gives $\pi_0 = e^{-\rho}$ and generally,

$$\pi_n = e^{-\rho} \cdot \frac{\rho^n}{n!}, \quad n = 0, 1, 2, ...$$

which is a Poisson distribution with mean ρ.

5. At state 0, we have $\gamma_{0k} = \lambda p_k$ for $k \geq 1$ and to get row sum zero, we get $\gamma_{00} = -\lambda$. As $\gamma_{k0} = \mu$ for all k, the first column in G is $(-\lambda, \mu, \mu, ...)$ and the first equation in $\pi G = 0$ becomes

$$-\lambda\pi_0 + \mu(\pi_1 + \pi_2 + ...) = -\lambda\pi_0 + \mu(1 - \pi_0) = 0$$

which gives

$$\pi_0 = \frac{\mu}{\lambda + \mu} = \frac{1}{11}$$