Stochastic Processes, Solutions 3

1(10) Interpret "at random" as chosen according to the stationary distribution. The equation $\pi P = \pi$ is

$$(\pi_c \ \pi_v) \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix} = (\pi_c \ \pi_v)$$

which gives $\pi_c \approx 0.62, \pi_v \approx 0.38$.

(a) Guess c(onsonant); probability 0.62
(b) Guess c; probability 0.62 (stationary)
(c) Guess ccccc; probability 0.62^5
(d) Guess cvcvc; probability $0.62 \cdot 0.5 \cdot 0.8 \cdot 0.5 \cdot 0.8$

1(14)(a) 4 steps (states 0–3) **(b)** The period is 2
(c) Solving $\pi P = \pi$ gives $\pi = (1/5, 1/5, 1/5, 1/5, 1/10, 1/10)$
(d) No. For example, $p_{00}^{(n)} = 0$ for all odd n so $p_{00}^{(n)} \not\to \pi_0$

2(a) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 5, ...\}$ which gives $d(0) = 1$
(b) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 7, ...\}$ which gives $d(0) = 1$
(c) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 7, ...\}$ which gives $d(0) = 1$
(d) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 6, 8, 10, ...\}$ which gives $d(0) = 2$
(e) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 8, 12, ...\}$ which gives $d(0) = 4$
(f) $\{n \geq 1 : p_{00}^{(n)} > 0\} = \{4, 7, ...\}$ which gives $d(0) = 1$

Turn-in problems

1(a) The stationary distribution is given by

$$\pi_0 = \frac{1-2p}{2-2p}$$

and

$$\pi_k = \frac{1-2p}{2-2p} \frac{p^{k-1}}{(1-p)^k} \qquad k \geq 1$$

where $p = 0.49$ which gives

$$E_5[\tau_5] = \frac{1}{\pi_5} \approx 30.6$$

(b) With π as above we get

$$E_5[N_0] = \frac{\pi_0}{\pi_5} \approx 0.6$$

2(a) Since $Y_n = X_{2n}$ we get transition probabilities

$$P(Y_1 = j | Y_0 = i) = P(X_2 = j | X_0 = i) = p_{ij}^{(2)}$$

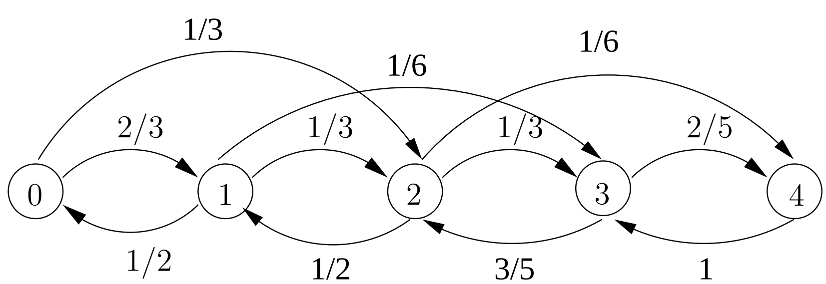
which is the (i, j)th entry in the matrix P^2. Hence, the Y-chain has transition matrix P^2.

(b) No. Let $S = \{0, 1\}$ and let $p_{01} = p_{10} = 1$. Then $\{X_n\}$ is irreducible but $\{Y_n\}$ is not because we cannot move from 0 to 1 or 1 to 0 in an even number of steps.

(c) Yes. Denote the transition probabilities of $\{Y_n\}$ by q_{ij} so that $q_{ij} = p_{ij}^{(2)}$. If $\{Y_n\}$ is irreducible, for each pair of states i and j, there exist m and n such that $q_{ij}^{(m)} > 0$ and $q_{ji}^{(n)} > 0$ which means that $p_{ij}^{(2m)} > 0$ and $p_{ji}^{(2n)} > 0$ and $\{X_n\}$ is irreducible.

In words, irreducibility of the X-chain means that all states can reach each other; irreducibility of the Y-chain means that all states can reach each other in an even number of steps, clearly a stronger statement.

3(a) The transition graph is



and the transition matrix is

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1/6 & 0 \\ 0 & 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 3/5 & 0 & 2/5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

To understand the transition probabilities $p_{32} = 3/5$ and $p_{34} = 2/5$, assume the chain is in state 3. On average, out of every 6 events, 3 are departures, 2 are arrivals of single individuals, and 1 is an arrival of a pair. In the last case nothing happens which leaves 5 events, 3 of which are departures, leading to state 2, and 2 of which are single arrivals, leading to state 4.

Formally, let A be the event that the chain jumps to state 4 and condition on the events B_1: departure, B_2: arrival of pair, and B_3: arrival of single individual. By LTP

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_3) + P(A|B_3)P(B_3)$$

$$= 0 \cdot \frac{1}{2} + P(A) \cdot \frac{1}{6} + 1 \cdot \frac{1}{3}$$

$$= P(A) \cdot \frac{1}{6} + \frac{1}{3}$$

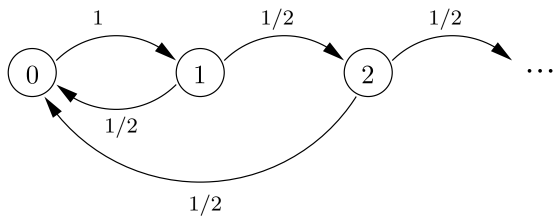
where $P(A|B_2) = P(A)$ because if a pair arrives, nothing happens in the system. Solve for $P(A)$ to get $P(A) = 2/5$.

Solving $\pi P = \pi$ gives $\pi = (1/10, 1/5, 4/15, 5/18, 7/45)$.

(b) Asssuming the system has reached stationarity, the probability is $\pi_0 = 1/10$.

(c) $E_0[N_4] = \dfrac{\pi_4}{\pi_0} = \dfrac{7/45}{1/10} \approx 1.6$

4. The transition graph is



and solving $\pi P = \pi$ gives stationary distribution $\pi_0 = 1/3, \pi_k = (1/3)(1/2)^{k-1}$ for $k \geq 1$. Note that π_0 is now smaller than in Example 7.2.16 because the chain always leaves state 0 immediately and thus spends less time there. Also note that the chain spends an equal amount of time in 0 and 1. As the chain is irreducible and aperiodic (0 can reach 0 in 2 or 3 steps), π is the limit distribution.