Stochastic Processes, Final Exam, Spring 2013

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1. If a service system has so many servers that virtually nobody ever has to wait in line, we can assume that the number of servers is infinite and get a queueing system denoted $M/M/\infty$. Sketch the rate diagram, find the generator, and decide when a stationary distribution exists and what it is expressed in terms of $\rho = \lambda/\mu$.

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2. Consider a bacterial population where individuals have lifetimes that are $\exp(\alpha)$. There is immigration into the population such that groups of immigrants arrive according to a Poisson process with rate λ. A group is of size k with probability p_k for $k \geq 1$. Individual bacteria reproduce by dividing where the probability that a bacterium in a population of size k divides rather than dies equals q_k for $k \geq 1$. As long as $k \leq 10^4$ there is no death, that is, $q_k = 1$ for $k \leq 10^4$. Occasionally, there is a disaster that kills the entire bacterial population. Suppose that disasters occur according to a Poisson process with rate $\mu = 0.1\lambda$. What is the probability that an arriving immigrant finds the population extinct?

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3. Find the extinction probabilities for the following branching processes.

(a) An individual has 0 offspring with probability 3/4 and 4 offspring with probability 1/4.

(b) A cell dies with probability 1/5 and divides with probability 4/5.

(c) The offspring distribution has pgf $G(s) = e^{(s-1)/3}$

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4. Consider a queueing system where arrivals occur according to a Poisson process with rate λ and service times are independent $\sim \exp(\lambda)$ (same λ as for arrivals). There are 3 servers and no waiting room (customers who arrive when all servers are busy are lost).

(a) Sketch the rate diagram for the continuous-time chain and the transition diagram for the jump chain.

(b) State the generator for the continuous-time chain and the transition matrix for the jump chain.

(c) Find the stationary distribution π for the continuous-time chain and the stationary distribution ν for the jump chain. Verify that ν_k is proportional to $\lambda(k) \cdot \pi_k$ for $k = 0, 1, 2, 3$.

(d) In a given 2-hour period, how many minutes can you expect the system to be empty?