Complex Variables
Spring 2011

Assignment 9.1
Due October 31

Exercise 1. Use Cauchy's Integral Formula to evaluate the following definite integrals. All contours are positively oriented.

a. $\displaystyle\int_\gamma \frac{\sin e^z}{z^3}\,dz$ where γ is any simple closed curve containing the origin.

b. $\displaystyle\int_\gamma \frac{z}{(z-a)^2}\,dz$ where γ is any simple closed curve containing the complex number a. How does your result compare with that of Exercise 9.1.1b?

c. $\displaystyle\int_\gamma \frac{e^{z^2}}{(z^2-2)^3}\,dz$ where γ is the circle $|z-1|=2$.

Exercise 2. Let f be entire and suppose that there is a positive constant M and an $n \in \mathbb{N}$ so that $|f(z)| \leq M|z|^n$ for all z.

a. Prove that $f^{(n+1)}$ is constant.

b. Use part (a) and induction on n to prove that f is a polynomial of degree $\leq n$.

Exercise 3. Let f be an entire function.

a. Show that if $\operatorname{Re} f$ is bounded above then f is constant. [*Suggestion:* Consider the function $e^{f(z)}$.]

b. Conclude that f is constant under any of the following conditions: $\operatorname{Re} f$ is bounded below; $\operatorname{Im} f$ is bounded above; $\operatorname{Im} f$ is bounded below.

Exercise 4. Let f be entire. Suppose there are $r > 0$ and $z_0 \in \mathbb{C}$ so that $f(\mathbb{C}) \cap D(z_0; r) = \varnothing$. Prove that f is constant. [*Suggestion:* Consider the function $1/(f(z) - z_0)$.] That is, unless f is constant, the image of an entire function gets arbitrarily close to every point in the complex plane.