Putnam Exam Seminar — Assignment 12

Fall 2012 — Due November 19

Exercise 1. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$. [Putnam 2002, A1]

Exercise 2. Let k be the smallest positive integer with the property that there are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved. [Putnam 1985, B1]

Exercise 3. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (*Note:* $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.) [Putnam 2005, B1]

Exercise 4. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n + 1) f_n(x + 1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes. [Putnam 1985, B2]