Putnam Exam Seminar
Fall 2012 September 10

In its simplest form, the *pigeonhole principle* (also known as *Dirichlet's box principle*) has the following statement.

Pigeonhole principle, version 1. *If* $n+1$ *objects are distributed among* n *boxes, then one of the boxes will contain at least* 2 *objects.*

A somewhat stronger version is the following.

Pigeonhole principle, version 2. *If* $kn+1$ *objects are distributed among* n *boxes, then one of the boxes will contain at least* $k+1$ *objects.*

Note that the first version follows from the second by setting $k = 1$. However, the pigeonhole principle is rarely used in precisely this form, since one may not always be trying to distribute a number of objects that is only 1 more than a multiple of the number of boxes. Therefore the following even more general version can sometimes be helpful.

Pigeonhole principle, version 3. *If* m *objects are distributed among* n *boxes, then one of the boxes will contain at least* $\lceil m/n \rceil$ *objects.*

Here $\lceil \cdot \rceil$ denotes the *ceiling function*, i.e. $\lceil x \rceil$ is the least integer greater than or equal to x. It is not hard to see that this version implies the second version. It also implies the next result, which is perhaps the most commonly encountered version of the pigeonhole principle.

Pigeonhole principle, version 4. *If* m *objects are distributed among fewer than* m *boxes, then one of the boxes will contain at least 2 objects.*

Exercise 1. Given a set of $n+1$ positive integers, none of which is greater than $2n$, prove that at least one member of this set must divide another member.

Exercise 2. Prove that if any five points are chosen on a sphere, then four of them lie on some closed hemisphere.

Exercise 3. Prove that every set of 10 two-digit positive integers has two disjoint subsets with the same sum of elements.