Putnam Exam Seminar — Quiz 10
Fall 2012 — November 7

Problem 1. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges. [Putnam 1994, A1]

Problem 2. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges. [Putnam 1988, A3]

Problem 3. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min \left(\left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right)$$

(Here $\min(a, b)$ denotes the minimum of a and b.) [Putnam 1997, B1]

Problem 4. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

[Putnam 2001, B3]