Putnam Exam Seminar | Quiz 5
Fall 2012 | October 3

Problem 1. Functions f, g and h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \; f(0) = 1,$$
$$g' = fg^2h + \frac{4}{fh}, \; g(0) = 1,$$
$$h' = 3fgh^2 + \frac{1}{fg}, \; h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0. [Putnam 2009, A2]

Problem 2. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x?

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x?

[Putnam 1987, A3]

Problem 3. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine whether there exists a nonzero function defined on an open interval (a, b) such that this wrong product rule is true for x in (a, b). [Putnam 1988, A2]

Problem 4. If

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots \,,$$
$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots \,,$$
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots \,,$$

prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

[Putnam 1939, 14]