Putnam Exam Seminar — Quiz 7
Fall 2012 — October17

Problem 1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A *perfect square* is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. We say that a is within n of b if $b - n \le a \le b + n$.) [Putnam 1994, B1]

Problem 2. What is the units (i.e. rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$, where $[x]$ denotes the greatest integer which is less than or equal to x? [Putnam 1986, A2]

Problem 3. Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of the squares of two integers. For example, $n = 0$ is such an integer, since $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$ and $2 = 1^2 + 1^2$. [Putnam 2000, A2]

Problem 4. Let A denote the sum of the decimal digits of 4444^{4444} and let B be the sum of the decimal digits of A. Find the sum of the decimal digits of B. [Int. Olympiad 1975]

Problem 5. Prove that, for any positive integer n,

$$1^n + 2^n + 3^n + 4^n$$

is divisible by 5 if and only if n is *not* divisible by 4. [Hungary 1901]

Problem 6. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

[Putnam Exam 1991, B-4]