Flux Integrals: Stokes' and Gauss' Theorems

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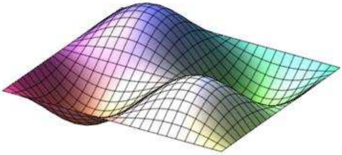
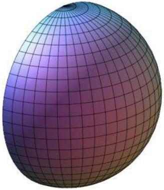
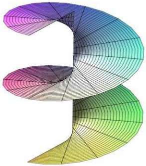
Calculus III
December 4, 2012

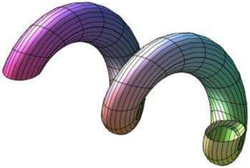
Surfaces

A *surface* S is a subset of $\mathbb{R}^3$ that is "locally planar," i.e. when we zoom in on any point $P \in S$, S looks like a piece of a plane.





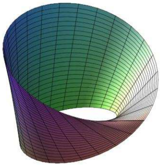




Orientable surfaces

A surface S is *orientable* if it is "two sided."

- Every surface shown above is orientable.
- The *Möbius band* is *not* orientable.



If S is an oriented surface, an *orientation* of S is a choice of a particular side of S as "positive."

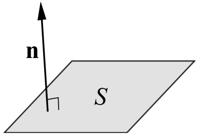
Planar flux

If S is an oriented (finite) part of a plane and $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a constant vector field, the *flux of* $\mathbf{F}$ *through* S is defined to be

$$\text{comp}_{\mathbf{n}}(\mathbf{F})\, A(S)$$

where:

- $\mathbf{n}$ is the normal vector to S, in the "positive" direction;
- $A(S)$ is the area of S.



If F represents the "flow" of some quantity, then the flux is the amount of "stuff" that passes through S in one unit of time.

General flux

Suppose S is a more general oriented surface, and $\mathbf{F} = \mathbf{F}(x, y, z)$ is a possibly nonconstant vector field.

- Subdivide S into (approximately) planar pieces with (inherited) normal vectors $\mathbf{n}_j$ and areas ΔS_j.
- Choose a point P_j in the jth subdivision, and assume that $\mathbf{F} \approx \mathbf{F}(P_j)$ throughout this subdivision.
- Compute the "local planar flux" on each subdivision and add to get the total approximate flux:

$$\sum_j \mathrm{comp}_{\mathbf{n_j}}(\mathbf{F}(P_j))\,\Delta S_j.$$

- Take the limit as the areas of the subdivisions tend to zero to get the *flux of* $\mathbf{F}$ *through* S:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \lim_{\Delta S \to 0} \sum_j \mathrm{comp}_{\mathbf{n_j}}(\mathbf{F}(P_j))\,\Delta S_j.$$

Remarks

- If we assume that $\mathbf{n_j}$ is a unit vector, then

$$\begin{aligned}\text{comp}_{\mathbf{n}_j}(\mathbf{F}(P_j))\,\Delta S_j &= (\mathbf{F}(P_j)\cdot\mathbf{n}_j)\,\Delta S_j\\ &= \mathbf{F}(P_j)\cdot(\Delta S_j\,\mathbf{n}_j)\\ &= \mathbf{F}(P_j)\cdot\Delta\mathbf{S}_j,\end{aligned}$$

  where $\Delta\mathbf{S_j}$ is a normal vector with area ΔS_j.

  The $d\mathbf{S}$ is thus meant to represent an "infinitesimal area normal vector" to S.

- As with planar flux, if $\mathbf{F}$ represents the "flow" of some quantity, then $\iint_S \mathbf{F}\cdot d\mathbf{S}$ represents the amount of "stuff" that passes through S in one unit of time.

Computing flux integrals

In order to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, one must first parametrize S via a two-variable vector function:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \ \ (u,v) \in D \subset \mathbb{R}^2.$$

If we define

$$\mathbf{T}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k},$$
$$\mathbf{T}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

then $\mathbf{T}_u \times \mathbf{T}_v$ is normal to S at every point. If the direction of $\mathbf{n}$ agrees with the orientation of S, a Riemann sum argument shows

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv.$$

Example

Find the flux of the vector field

$$\mathbf{F} = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$$

through the portion of the sphere $x^2 + y^2 + z^2 = 4$ *in the first octant, oriented toward the origin.*

The portion of the sphere in question can be parametrized as

$$\mathbf{r}(u, v) = 2\sin u \cos v\,\mathbf{i} + 2\sin u \sin v\,\mathbf{j} + 2\cos u\,\mathbf{k},$$
$$0 \le u \le \pi/2, \;\; 0 \le v \le \pi/2.$$

The tangent vectors are

$$\mathbf{T}_u = 2\cos u \cos v\,\mathbf{i} + 2\cos u \sin v\,\mathbf{j} - 2\sin u\,\mathbf{k},$$
$$\mathbf{T}_v = -2\sin u \sin v\,\mathbf{i} + 2\sin u \cos v\,\mathbf{j}.$$

We have

$$\mathbf{T}_u \times \mathbf{T}_v = 4\sin^2 u \cos v\,\mathbf{i} + 4\sin^2 u \sin v\,\mathbf{j} + 4\sin u \cos u\,\mathbf{k}$$

and

$$\mathbf{F}(\mathbf{r}(u,v)) = 2\sin u \cos v\,\mathbf{i} - 2\cos u\,\mathbf{j} + 2\sin u \sin v\,\mathbf{k},$$

so that

$$\mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) = 8\sin^3 u \cos^2 v.$$

Since $\mathbf{T}_u \times \mathbf{T}_v$ is oriented *outward* we have

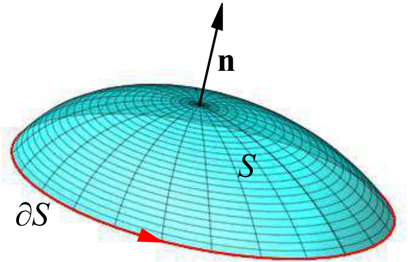
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\int_0^{\pi/2}\int_0^{\pi/2} 8\sin^3 u \cos^2 v\,du\,dv = -\frac{4\pi}{3}$$

A relationship between surface and line integrals

Stokes' Theorem

Let S be an oriented surface bounded by a closed curve ∂S. If $\mathbf{F}$ is a C^1 vector field and ∂S is oriented positively relative to S, then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$



Remarks

- Stokes' Theorem is another generalization of FTOC. It relates the integral of "the derivative" of **F** on S to the integral of **F** itself on the boundary of S.

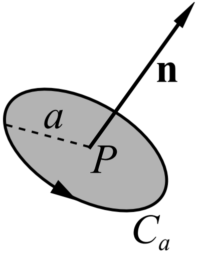
- If $D \subset \mathbb{R}^2$ is a 2D region (oriented upward) and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a 2D vector field, one can show that

$$\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

  That is, Stokes' Theorem *includes* Green's Theorem as a special case.

Interpreting the curl

Let **F** be a vector field. Fix a point $P \in \mathbb{R}^3$ and a unit vector **n** based at P. Let C_a denote a circle of radius a, centered at P_0, in the plane normal to **n**, oriented using the right hand rule.



The tendency of **F** to "circulate" about **n** (in the positive sense) can be measured by

$$\lim_{a\to 0^+} \frac{1}{\pi a^2} \int_{C_a} \mathbf{F}\cdot d\mathbf{r}.$$

If D_a is the disk bounded by C_a, Stokes' Theorem implies

$$\begin{aligned}\lim_{a\to 0^+}\frac{1}{\pi a^2}\int_{C_a}\mathbf{F}\cdot d\mathbf{r} &= \lim_{a\to 0^+}\frac{1}{\pi a^2}\iint_{D_a}\nabla\times\mathbf{F}\cdot d\mathbf{S}\\ &= \lim_{a\to 0^+}\frac{1}{\pi a^2}(\nabla\times\mathbf{F})(P_0)\cdot\mathbf{n}\,A(D_a)\\ &= (\nabla\times\mathbf{F})(P_0)\cdot\mathbf{n}.\end{aligned}$$

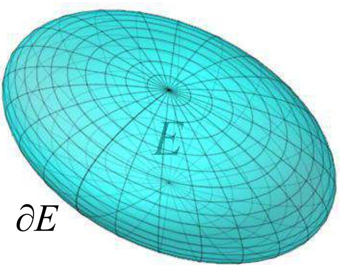
Thus, the circulation at P about $\mathbf{n}$ is *maximized when* $\mathbf{n}$ *points in the same direction as* $\nabla\times\mathbf{F}$*.*

A relationship between surface and triple integrals

Gauss' Theorem (a.k.a. The Divergence Theorem)

Let $E \subset \mathbb{R}^3$ *be a solid region bounded by a surface* ∂E. *If* $\mathbf{F}$ *is a* C^1 *vector field and* ∂E *is oriented outward relative to* E, *then*

$$\iiint_E \nabla \cdot \mathbf{F}\, dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}.$$



Remarks

- This can be viewed as yet another generalization of FTOC.
- Gauss' Theorem reduces computing the flux of a vector field through a *closed surface* to integrating its divergence over the region contained by that surface.
- As above, this can be used to derive a physical interpretation of $\nabla \cdot \mathbf{F}$:

$$(\nabla \cdot \mathbf{F})(P) = \lim_{a \to 0^+} \frac{1}{V(B_a)} \iiint_{B_a} \nabla \cdot \mathbf{F} \, dV$$
$$= \lim_{a \to 0^+} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S},$$

where $P \in \mathbb{R}^3$, B_a and S_a are the solid ball and sphere (respectively) of radius a centered at P.

A vast generalization

- We have studied various types of differentiation and integration in 2 and 3 dimensions.
- These can be generalized to arbitrary dimension n using the notions of "manifold" and "differential form."
- The following theorem unifies and extends much of our integration theory in one statement.

Generalized Stokes Theorem

If M is an n-dimensional "manifold with boundary," and ω is a "differential $(n-1)$-form," then

$$\int_M d\omega = \int_{\partial M} \omega.$$