Number Theory II
Fall 2012

Assignment 1
Due September 4

Exercise 1. Fix a real number $x > 2$. Let $p \leq x$ be a prime number. Prove that there is a natural number n_p so that

$$p^{n_p} \leq x < p^{n_p+1}.$$

Exercise 2. Consider the x from the preceding exercise. Let

$$\mathcal{P} = \{p \leq x \,|\, p \text{ is prime}\},$$
$$\mathcal{N} = \{p_1^{a_1} p_2^{a_n} \cdots p_r^{a_r} \,|\, r \geq 0, p_i \in \mathcal{P}, 1 \leq a_i \leq n_{p_i}\}.$$

Show that if $n \leq x$ is a positive integer, then $n \in \mathcal{N}$. Does $\mathcal{N} = \{n \leq x\}$?

Exercise 3. Let x be as above. Use the preceding exercise to show that

$$\prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{n_p}}\right) \geq \sum_{n \leq x} \frac{1}{n}.$$

Exercise 4. With x as above, show that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \geq \sum_{n \leq x} \frac{1}{n}.$$

Exercise 5. Prove that

$$\lim_{x \to \infty} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \infty.$$

Conclude that there are infinitely many prime numbers.