Number Theory II | Assignment 3.2
Fall 2012 | Due September 18

Exercise 1. Interpret, and then verify, the following equalities. Throughout, $g(x)$ and $h(x)$ denote nonnegative functions.

a. $O(g(x)) + O(h(x)) = O(\max\{g(x), h(x)\})$.

b. $g(x)O(h(x)) = O(g(x))\,O(h(x)) = O(g(x)h(x))$.

c. $O(O(g(x))) = O(g(x))$.

Exercise 2. Let $f(x)$ and $g(x)$ be polynomials with real coefficients of degrees m and n, respectively.

a. Prove that
$$\lim_{x\to\infty} \frac{f(x)x^{n-m}}{g(x)}$$
exists and is nonzero. [*Suggestion:* Factor the largest power of x out of both the numerator and denominator.]

b. Prove that
$$\frac{f(x)}{g(x)} = O\left(x^{m-n}\right)$$
for all sufficiently large x. In particular, if $g(x)$ has no real roots, show this holds for $x \geq 1$.

c. What can you conclude about $f(x)$ if you take $g(x) = 1$? Does this seems reasonable?

Exercise 3. Let $\epsilon > 0$.

a. Show that
$$\lim_{x\to\infty} \frac{\log x}{x^\epsilon} = 0.$$

b. Explain why $\log x / x^\epsilon$ is bounded on any interval of the form $[2, b]$.

c. Prove that $\log x = O(x^\epsilon)$ for $x \geq 2$. Does the implied constant depend on ϵ?