PUTNAM EXAM SEMINAR
FALL 2013

ASSIGNMENT 10
DUE NOVEMBER 25

Exercise 1. Determine whether or not the matrix

$$\begin{pmatrix} 117 & 218 & 344 & 511 & 1007 \\ 101 & 800 & 911 & 578 & 113 \\ 1212 & 14 & 4216 & 178 & 2013 \\ 516 & 19 & 2114 & 104 & 3416 \\ 789 & 534 & 114 & 472 & 300 \end{pmatrix}$$

has an inverse.

Exercise 2. Prove that $\dfrac{x^5}{5} + \dfrac{x^3}{3} + \dfrac{7x}{15}$ is an integer for every integral value of x.

Exercise 3. Prove that, for any positive integer n,

$$1^n + 2^n + 3^n + 4^n$$

is divisible by 5 if and only if n is *not* divisible by 4. [Hungary 1901]

Exercise 4. Consider the set $\{2, 5, 13\}$. Show that if $D \notin \{2, 5, 13\}$, then there exist $A, B \in \{2, 5, 13, D\}$ so that $AB - 1$ is not a perfect square.

Exercise 5. Prove that every positive integer has a multiple whose decimal representation includes all ten digits. [Putnam 1956, 2]

Exercise 6. Let A denote the sum of the decimal digits of 4444^{4444} and let B be the sum of the decimal digits of A. Find the sum of the decimal digits of B. [Int. Olympiad 1975]

Exercise 7. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A *perfect square* is the square of an integer. We say that a is within n of b if $b - n \leq a \leq b + n$.) [Putnam 1994, B1]