Putnam Exam Seminar — Fall 2013

Assignment 5 — Due October 14

Exercise 1. Find all real-valued continuously differentiable functions f defined on the real line such that for all x

$$(f(x))^2 = 1990 + \int_0^x [(f(t))^2 + (f'(t))^2]\, dt.$$

[Putnam 1990, B1]

Exercise 2. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers (x, y),

$$f(x+y) = f(x)f(y) - g(x)g(y)$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x. [Putnam 1991, B2]

Exercise 3. Prove that $f(n) = 1 - n$ is the only integer-valued function on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;

(ii) $f(f(n+2)+2) = n$ for all integers n;

(iii) $f(0) = 1$.

[Putnam 1992, A1]

Exercise 4. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded. [Putnam 1997, B2]