Putnam Exam Seminar
Fall 2013

Assignment 6
Due October 21

**Exercise 1.** A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval $(a, b)$ and a non-zero function $g$ defined on $(a, b)$ such that this wrong product rule is true for $x$ in $(a, b)$. [Putnam 1988, A2]

**Exercise 2.** Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers $(x, y)$,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$. [Putnam 1991, B2]

**Exercise 3.** Suppose the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h = a\frac{\partial h}{\partial x} + b\frac{\partial h}{\partial y}$$

for some constants $a$ and $b$. Prove that if there is a constant $M$ such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then $h$ is identically zero. [Putnam 2010, A3]