Putnam Exam Seminar — Quiz 6
Fall 2013 — October 30

Exercise 1. If

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots ,$$
$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots ,$$
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots ,$$

the prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

[Putnam 1939, 14]

Exercise 2. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

[Putnam 1999, A3]

Exercise 3. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

[Putnam 2001, B3]