# SECOND ORDER LINEAR ODES

A homogeneous second order linear differential equation with constant coefficients has the form

$$ay'' + by' + cy = 0 \tag{1}$$

where $a, b, c$ are real constants and $a \neq 0$. As we learned in class, central to the study of such equations is the following version of the *Principle of Superposition*.

**Theorem 1.** *If $y_1$ and $y_2$ are any two solutions to (1), then so is*

$$y = c_1y_1 + c_2y_2, \tag{2}$$

*where $c_1$ and $c_2$ are arbitrary real numbers. If, moreover, $y_1$ and $y_2$ are* linearly independent *(i.e. not proportional), then (2) gives a complete description of the solution set to (1).*

This result reduced the problem of finding the general solution to (1) to the (perhaps!) simpler problem of finding only *two* special (i.e. independent) solutions. To achieve this latter goal, we introduced the characteristic polynomial

$$ar^2 + br + c = 0 \tag{3}$$

and found that we can always use its roots to produce a pair of independent solutions $y_1$ and $y_2$ to (1), thereby obtaining the general solution $y = c_1y_1 + c_2y_2$.

While this provides us with a very convenient and efficient means of finding the general solution to a differential equation of the form (1), it is also somewhat unsatisfactory because we've made no attempt to justify Theorem 1. There's a very good reason for this. While it isn't hard to check that the functions $y$ described in Theorem 1 are indeed solutions to (1), checking that *every* solution actually has this form is much more difficult. This should be contrasted with the procedures that we have for solving first order separable or linear equations. In these cases the completeness of our solutions is immediate, since they are obtained from a step by step algebraic manipulation of the original differential equation.

The goal of this note is to bridge this gap in our knowledge by proving Theorem 1. This we will do by establishing the somewhat weaker result stated below.

**Theorem 2.** *If the characteristic equation (3) has two distinct real roots $r_1$ and $r_2$ then the general solution to (1) is given by*

$$y = c_1e^{r_1x} + c_2e^{r_2x} \tag{4}$$

*where $c_1$ and $c_2$ are arbitrary constants. If the characteristic equation (3) has only a single (repeated) real root $r$ then the general solution to (1) is given by*

$$y = (c_1x + c_2)e^{rx} \tag{5}$$

*where* $c_1$ *and* $c_2$ *are arbitrary constants. And if the characteristic equation* (3) *has nonreal (complex) roots* $\alpha \pm \beta i$ *then the general solution to* (1) *is given by*

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \tag{6}$$

*where* $c_1$ *and* $c_2$ *are arbitrary constants.*

Note that (somewhat ironically), Theorem 2 provides us with an explicit description of the general solution to (1), precluding (from a purely computational standpoint) the need for Theorem 1 in the first place. As such, anyone who wishes may skip the later derivation of Theorem 1 from Theorem 2. We should also point out that in the course of the proof of Theorem 2 we will find a deeper explanation for the specific forms of the solutions to (1). Certainly better than "Well, just plug them in and notice that they happen to work," which is as much justification as we got in class.

*Proof of Theorem 2.* It is easy (but perhaps a bit tedious) to check that the functions given in (4) to (6) do, indeed, yield solutions to (1) in each case. The question is: are these the only solutions? We show in every case that they are.

We can deal with the first two cases simultaneously. Suppose that $y$ is a solution to (1) and that $r_1$ is a real root of (3). Define a new function $u$ by

$$u = ay' + (ar_1 + b)y. \tag{7}$$

Since $y$ and $y'$ are differentiable so is $u$, and in fact

$$u' = ay'' + (ar_1 + b)y'.$$

Therefore

$$\begin{aligned} u' - r_1 u &= ay'' + (ar_1 + b)y' - r_1 ay' - r_1(ar_1 + b)y \\ &= ay'' + by' - (ar_1^2 + br_1)y. \end{aligned}$$

However, we know that $ar_1^2 + br_1 + c = 0$ so that $ar_1^2 + br_1 = -c$. That is

$$u' - r_1 u = ay'' + by' - (ar_1^2 + br_1)y = ay'' + by' + cy = 0.$$

Therefore, $u$ is a solution to the *first order* linear differential equation $u' - r_1 u = 0$! Solving this using our earlier techniques we find that we must have $u = C_1 e^{r_1 x}$ for some constant $C_1$. If we put this back into (7) we find that

$$ay' + (ar_1 + b)y = C_1 e^{r_1 x}$$

which is another first order linear equation, this time in $y$! Dividing by $a$ (which we know to be nonzero) it becomes

$$y' + \left(r_1 + \frac{b}{a}\right) y = C_2 e^{r_1 x} \tag{8}$$

where $C_2 = C_1/a$.

If $r_2$ is the other root of $ar^2 + br + c = 0$ (which may, in fact, equal $r_1$), then the polynomial $ax^2 + bx + c$ must factor as $a(x - r_1)(x - r_2)$. If we multiply out the latter polynomial and compare its $x$ coefficient with the first, we find that $b = -a(r_1 + r_2)$, or $-r_2 = r_1 + b/a$.[1] So we can rewrite (8) as

$$y' - r_2 y = C_2 e^{r_1 x}.$$

Using the integrating factor $e^{-r_2 x}$ and solving we find that $y$ must be given by

$$y = C_2 e^{r_2 x} \int e^{(r_1 - r_2)x}\, dx.$$

There are now two cases. If $r_1 \neq r_2$ (i.e. the characteristic equation has two distinct real solutions) then $r_1 - r_2 \neq 0$ and so we have

$$\int e^{(r_1-r_2)x}\, dx = \frac{e^{(r_1-r_2)x}}{r_1 - r_2} + C_3 \tag{9}$$

which means that

$$y = C_2 e^{r_2 x}\left(\frac{e^{(r_1-r_2)x}}{r_1 - r_2} + C_3\right) = C_4 e^{r_1 x} + C_3 e^{r_2 x},$$

where we have set $C_4 = C_2/(r_1 - r_2)$. Up to the names of the constants (which are arbitrary anyway), this is exactly what we needed! But what if $r_1 = r_2$ so that (9) isn't valid? In this case we have

$$y = C_2 e^{r_2 x} \int e^{(r_1-r_2)x}\, dx = C_2 e^{r_1 x} \int dx = C_2 e^{r_1 x}(x + C_3) = (C_2 x + C_4)e^{r_1 x},$$

with $C_4 = C_2 C_3$, which gives us what we expected in this case, too.

Now let's move on to the third case, in which the characteristic equation (3) has nonreal roots $\alpha \pm \beta i$. Since we're after *real*-valued solutions in a situation which more naturally calls for complex numbers, things are a bit more complicated. The first thing we need is a relationship between $\alpha$, $\beta$ and the coefficients of (3). As above, knowing the roots of a polynomial allows us to factor it, so that $ax^2 + bx + c = a(x - (\alpha + \beta i))(x - (\alpha - \beta i)) = a(x^2 - 2\alpha x + (\alpha^2 + \beta^2))$. Comparing coefficients in these expressions tells us that we must have

$$\begin{aligned} b &= -2a\alpha && (10) \\ c &= a(\alpha^2 + \beta^2). && (11) \end{aligned}$$

We'll need these relationships shortly.

Now assume that $y$ is a solution to (1). We're going to perform a series of changes of variables to get (1) into a friendlier form. We first set

$$u = e^{-\alpha x} y. \tag{12}$$

Differentiating twice yields

$$u'' = e^{-\alpha x}\left(y'' - 2\alpha y' + \alpha^2 y\right)$$

[1] This can also be seen directly by appealing to the quadratic formula, which expresses $r_1$ and $r_2$ in terms of $a$, $b$ and $c$.

so that

$$\begin{aligned} au'' + \beta^2 au &= e^{-\alpha x}\left(ay'' - 2a\alpha y' + a(\alpha^2+\beta^2)y\right) \\ &= e^{-\alpha x}\left(ay'' + by' + cy\right) \\ &= 0 \end{aligned}$$

where we have used (10), (11) and the fact that $y$ solves (1). If we divide both sides of this equation by $a$, which we know to be nonzero, we obtain the simple linear equation

$$u'' + \beta^2 u = 0. \tag{13}$$

At this point it's worth remembering that our independent variable has been assumed to be $x$. We now make the substitution $t = \beta x$. According to the chain rule

$$\begin{aligned} u' &= \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \beta\frac{du}{dt} \\ u'' &= \frac{d^2u}{dx} = \frac{d}{dx}\frac{du}{dx} = \frac{d}{dx}\left(\beta\frac{du}{dt}\right) = \beta\frac{d^2u}{dt^2}\frac{dt}{dx} = \beta^2\frac{d^2u}{dt^2}. \end{aligned}$$

Therefore (13) becomes

$$\beta^2\frac{d^2u}{dt^2} + \beta^2 u = 0.$$

Since we know $\alpha + \beta i$ is definitely not real it must be that $\beta \neq 0$. We can thus divide both sides of the equation above by $\beta^2$ which tells us that

$$\frac{d^2u}{dt^2} + u = 0. \tag{14}$$

To solve (14) for $u$ we perform one final substitution, letting

$$w = \frac{du}{dt} + (\tan t)u. \tag{15}$$

Because $\tan t$ is only defined on intervals of the form $I_n = (n\pi - \pi/2, n\pi + \pi/2)$, where $n$ is an integer, $w$ is only defined on these intervals. So, from this point on let's assume that our $t$ domain is a single $I_n$. We find that

$$\begin{aligned} \frac{dw}{dt} - (\tan t)w &= \frac{d^2u}{dt^2} + (\sec^2 t)u + \tan t\frac{du}{dt} - \tan t\frac{du}{dt} - (\tan^2 t)u \\ &= \frac{d^2u}{dt^2} + (\sec^2 t - \tan^2 t)u \\ &= \frac{d^2u}{dt^2} + u \\ &= 0. \end{aligned}$$

That is, $w$ satisfies the linear equation $dw/dt - (\tan t)w = 0$, which is solved easily using the integrating factor $\cos t$. In fact

$$w = C_1 \sec t \tag{16}$$

for some constant $C_1$. Referring back to (15), this means that

$$\frac{du}{dt} + (\tan t)u = C_1 \sec t,$$

which is once again linear. The integrating factor $\sec t$ allows us to finally obtain

$$u = C_1 \sin t + C_2 \cos t$$

for some constant $C_2$.

At this point the back substitutions $t = \beta x$ and $y = e^{\alpha x} u$ tell us immediately that

$$y = e^{\alpha x} \left( C_2 \cos \beta x + C_1 \sin \beta x \right), \tag{17}$$

and we're finished. Well, not quite. There's still one technical point we need to address. When we defined $w$ in the previous paragraph we had to assume that its independent variable $t$ was restricted to the interval $I_n$. This means that the expression for $y$ in (17) is only valid on these intervals, and that in principle the constants $C_1$ and $C_2$ might vary as we vary $n$. However, $u$ and $du/dt$ are continuous (since they are both differentiable), and we can take limits at the endpoints of each interval to find that the constants match everywhere. For example, if we have

$$\begin{aligned} u &= A_1 \cos t + B_1 \sin t \text{ for } t \in I_0 \\ u &= A_2 \cos t + B_2 \sin t \text{ for } t \in I_1 \end{aligned}$$

then

$$B_1 = \lim_{t \to \pi/2^-} (A_1 \cos t + B_1 \sin t) = u\left(\frac{\pi}{2}\right) = \lim_{t \to \pi/2^+} (A_2 \cos t + B_2 \sin t) = B_2$$

and a similar computation with $du/dt$ gives $A_1 = A_2$. The general case is left to the reader. This completes the proof of the third case. □

The method of proof we've just employed actually still works (in principle) if we modify (1) so that it is no longer homogeneous. That is, if we replace (1) with

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is nonzero, the substitution $u = ay' + (ar_1 + b)y$ of the proof (in the case of real roots) yields the first order differential equation

$$u' - r_1 u = G(x)$$

for $u$, which is simply the inhomogeneous version of $u' - r_1 u = 0$. This equation is still linear, and if we solve it we can back substitute into $u = ay' + (ar_1 + b)y$ and solve for $y$. The reader is encouraged to give this a try in the cases where $G(x)$ is a constant or an exponential function.

*Proof of Theorem 1.* According to Theorem 2, the solutions to (1) are given by $y = c_1 u_1 + c_2 u_2$ for some pair of independent functions $u_1$, $u_2$. For example, in the case that the characteristic equation has two real roots $r_1$ and $r_2$ we have $u_1 = e^{r_1 x}$ and $u_2 = e^{r_2 x}$. Let $y_1$

and $y_2$ be any other pair of independent solutions. Then there are constants $a$, $b$, $c$ and $d$ so that $y_1 = au_1 + bu_2$ and $y_2 = cu_1 + du_2$. Because $y_1$ and $y_2$ are linearly independent, we can invert this relationship to obtain $u_1 = Ay_1 + By_2$ and $u_2 = Cy_1 + Dy_2$ for some $A$, $B$, $C$ and $D$. Thus, if $y = c_1u_1 + c_2u_2$ is any solution to (1), then

$$y = c_1(Ay_1 + By_2) + c_2(Cy_1 + Dy_2) = (c_1A + c_2C)y_1 + (c_1B + c_2D)y_2 = d_1y_1 + d_2y_2$$

for some constants $d_1$ and $d_2$. Since $y$ was an arbitrary solution to (1), this shows that *every* solution has the form stated in Theorem 1.

$\square$