Row Echelon Form

Definition: A matrix is in **(row) echelon form** provided:

1. All nonzero rows are above any rows of all zeros.
2. The leading entry of any row occurs to the right of the leading entry of the row above it.
3. All entries below a leading entry are zero.

Examples:

$$\begin{pmatrix} 5 & 1 & -6 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -6 \end{pmatrix} \qquad \begin{pmatrix} 0 & -3 & 3 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Some Generic Echelon Forms

$$\begin{pmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & \square \end{pmatrix} \qquad \begin{pmatrix} 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here $\square$ = any nonzero number, $*$ = any number.

Reduced Row Echelon Form

Definition: A matrix is in **reduced (row) echelon form** provided it is in (row) echelon form and also:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Examples:

$$\begin{pmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -7/5 \end{pmatrix}$$

Some Generic Reduced Echelon Forms

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Part 1: Get to echelon form

1. Locate leftmost nonzero column. This a **pivot column**.
2. If necessary, interchange rows to put (some) nonzero entry at the top of pivot column. This entry is now a **pivot position**.

$$\begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

3. Use replacements to create zeros below the pivot.

$$\begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

The replacements used here are:

$$R1 + R2 \mapsto R2$$
$$2 \cdot R1 + R3 \mapsto R3.$$

4. Ignore all rows above and including pivot. Repeat steps 1 - 3 on remaining matrix, if possible.

$$\left(\begin{array}{ccccc} \boxed{1} & 4 & 5 & -9 & -7 \\ \hline 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{\text{-5}} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This brings us to echelon form, usually not reduced.

Part 2: Continue to reduced echelon form

5. Working from right to left, use replacements to create zeros above each pivot. Use scaling to make each pivot a 1.

$$\begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{\text{-5}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & 0 & -7 \\ 0 & \boxed{2} & 4 & 0 & -6 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & 0 & -7 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 4 & 5 & 0 & -7 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & -3 & 0 & 5 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And we're done! Columns/rows containing pivots are called **pivot columns/rows**.