LINEAR ALGEBRA
FALL 2013

EXAM 2 REVIEW

Exercise 1. Let V be a vector space and let $H \subset V$ be a subspace. Show that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in H, then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is contained in H. Can you generalize this statement?

Exercise 2. Let V be a vector space and let $H, K \subset V$ be subspaces. The *intersection* of H and K, denoted $H \cap K$, is the collection of all vectors that belong to both H and K simultaneously. In set notation

$$H \cap K = \{\mathbf{v} : \mathbf{v} \text{ is in both } H \text{ and } K\}.$$

Show that $H \cap K$ is a subspace of V.

Exercise 3. Let V be a vector space and let $H, K \subset V$ be subspaces. The *sum* of H and K, denoted $H + K$, is the collection of all vectors of the form $\mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in H$ and $\mathbf{v} \in K$. In set notation

$$H + K = \{\mathbf{w} : \mathbf{w} = \mathbf{u} + \mathbf{v},\ \mathbf{u} \in H,\ \mathbf{v} \in K\}.$$

Show that $H + K$ is a subspace of V.

Exercise 4. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Let H be a subspace of W and let $T^{-1}(H)$ denote the set of all vectors $\mathbf{v} \in V$ so that $T(\mathbf{v}) \in H$. In set notation

$$T^{-1}(H) = \{\mathbf{v} \in V : T(\mathbf{v}) \in H\}.$$

Show that $T^{-1}(H)$ is a subspace of V.[1]

Exercise 5. Let V and W be vector spaces, $T : V \to W$ be a linear transformation and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a subset of V. Show that if $T(S) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ is linearly independent then S is also linearly independent.

Exercise 6. Let H and K be subspaces of a vector space V, and suppose that

$$\begin{aligned} H &= \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}, \\ K &= \text{Span}\ \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}. \end{aligned}$$

Show that $H + K = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$.

[1]The notation $T^{-1}(H)$ is *purely symbolic*. It *does not* mean that T is in invertible.

Exercise 7. Let $\mathcal{E}$ denote the standard basis for $\mathbb{R}^n$. Consider the linear transformation $Id : \mathbb{R}^n \to \mathbb{R}^n$ given by $Id(\mathbf{x}) = \mathbf{x}$. Find $[Id]_{\mathcal{B}}^{\mathcal{E}}$. Do you recognize this matrix?

Exercise 8. Let V be a vector space and let $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in V with the following property: given any vector $\mathbf{w} \in V$, the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{w}$ has *at most* one solution. Show that $\mathcal{C}$ must be a linearly independent set.

Exercise 9. Let

$$H = \left\{ \begin{pmatrix} a+c \\ a+b \\ a+4b-3c \\ a+3b-2c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

a. Show that H is a subspace of $\mathbb{R}^4$.

b. Find a basis for H.

Exercise 10. Let $\mathcal{B} = \{1, t-1, (t-1)^2\}$ and let $\mathcal{C} = \{1, t, t^2\}$

a. Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

c. Let $p(t) = -3t + t^2$. Compute $[p(t)]_{\mathcal{B}}$.

Exercise 11. Let $T : V \to W$ be a linear transformation between two vector spaces. Let $T(V)$ denote the range of T, which is a subspace of W. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V, show that $\{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ spans $T(V)$.

Exercise 12. Let

$$A = \begin{pmatrix} -3 & 11 & 9 & 4 & -1 \\ -4 & 3 & -2 & -2 & 0 \\ -2 & 4 & 2 & 2 & 4 \end{pmatrix}.$$

a. Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$.

b. Use your results from part (a) to determine if the map $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one or onto.

Exercise 13. Suppose that

$$A = \begin{pmatrix} 1 & 2 & b \\ 2 & a & c \\ 1 & 2 & d \\ 4 & 8 & e \end{pmatrix}$$

and that A can be row reduced to

$$U = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a. How are b, c, d, e related to the other entries in A?

b. What value(s) can a have?

c. Find a basis for Nul A.

Exercise 14. Show that the matrix

$$\begin{pmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{pmatrix}$$

is *not* invertible.

Exercise 15. Show that

$$N = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ c & 0 & a \end{pmatrix} \ : \ a, b, c \in \mathbb{R} \right\}$$

is a subspace of $M_{3\times 3}$.

Exercise 16. Let $\mathcal{B} = \{1, t, t^3\}$ and let $\mathcal{C} = \{5, 2-t, t^3 - 7t + 18\}$. Show that $\mathcal{B}$ and $\mathcal{C}$ span the same subspace of $\mathbb{P}_3$.