Math 22 Spring 2005

Linear Algebra with Applications

Final Exam

Saturday, June 4, 8 - 11 am, 101 Bradley

Your name (please print):

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted.** You must justify all of your answers to receive credit.

You have **three hours** to work on all **10** problems. Please do all of your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Problem 1.

a. Diagonalize the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}$$

if possible. If A cannot be diagonalized, explain why.

b. Find a number h so that the matrix

$$B = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is diagonalizable.

Problem 2. Let

$$H = \left\{ \begin{pmatrix} a+c \\ a+b \\ a+4b-3c \\ a+3b-2c \end{pmatrix} : a, b, c \text{ are real} \right\}.$$

a. Show that H is a subspace of $\mathbb{R}^4$.

b. Find a bases for H and $H^{\perp}$.

c. Find an *orthogonal* basis for H.

d. Let

$$\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Write $\mathbf{v}$ as the sum of a vector in H and a vector in $H^{\perp}$.

Problem 3.

a. Find the system of linear equations satisfied by a, b, c, d if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -4 & -5 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -4 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

b. Solve the system of equations found in part (a).

c. Let

$$A = \begin{pmatrix} -4 & -5 \\ 2 & -2 \end{pmatrix}$$

and let W be the set of all 2×2 matrices X so that $XA = AX$. Show that W is a subspace of $M_{2\times 2}$. Use your results from part (b) to find a basis for W.

Problem 4. Let

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -9 & -3 & 1 \end{pmatrix}.$$

a. Diagonalize A.

b. Let D be the diagonal matrix found in part (a). Find a diagonal matrix D' so that $(D')^2 = D$.

c. Use parts (a) and (b) together to find a matrix B so that $B^2 = A$. [*Hint:* B should be similar to D'.]

Problem 5. Determine h and k such that the system of equations

$$\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$$

has

a. no solution;

b. exactly one solution;

c. infinitely many solutions.

In cases (b) and (c), solve the system and write the solution in parametric (vector) form.

Problem 6. Let $\mathcal{B} = \{1, t-1, (t-1)^2\}$ and let $\mathcal{C} = \{1, t, t^2\}$

a. Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

c. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$. Then write $1 + t + t^2$ as a linear combination of the polynomials in $\mathcal{B}$.

Problem 7. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -10x_1 + 4x_2 \\ 11x_1 - 5x_2 \\ 20x_1 - 8x_2 \end{pmatrix}$$

a. Show that T is one-to-one but not onto.

b. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \end{pmatrix} \right\}, \mathcal{C} = \left\{ \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \right\}.$$

Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$.

c. Use part (b) and coordinates to show that the vector

$$\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

is *not* in the range of T (row reduction is not necessary).

Problem 8. Let

$$A = \begin{pmatrix} 4 & 1 & 3 & -1 & -1 \\ -1 & -5 & 0 & 0 & -2 \\ 5 & -3 & 1 & 4 & 3 \end{pmatrix}.$$

a. Find 3 columns of A that form an orthogonal set. Conclude that $\dim \operatorname{Col} A \geq 3$.

b. Use the Rank Theorem and part (a) to show that $\dim \operatorname{Nul} A = 2$.

Problem 9. Let A be an $n \times n$ matrix. Show that 1 is *not* an eigenvalue of A if and only if $A - I$ is invertible.

Problem 10. Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Let $T(V)$ denote the range of T, which we know is a subspace of W. Show that $\dim T(V) \leq \dim V$. [*Hint:* If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V, show that $\{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ spans $T(V)$.]

(Work Page)