PRACTICE PROBLEMS

Exercise 1. Find the values of c and d so that the matrix

$$B = \begin{pmatrix} 5 & 1 \\ c & d \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. [*Hint:* What conditions ensure that $\operatorname{Nul}(B - \lambda_1 I)$ and $\operatorname{Nul}(B - \lambda_2 I)$ are non-zero?]

Exercise 2. If a, b, c are real numbers with $a \neq c$, let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

a. Diagonalize A.

b. Compute the four entries of A^{1000} in terms of a, b, c.

Exercise 3.

a. Let W be a subspace of $\mathbb{R}^n$. Show that if $\mathbf{x} \in W$ and $\mathbf{x} \in W^\perp$ then $\mathbf{x} = \mathbf{0}$. [*Hint:* What is $\mathbf{x} \cdot \mathbf{x}$?]

b. Let A be an $m \times n$ matrix. Show that if $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} \in (\operatorname{Col} A)^\perp$. Use part (a) to conclude that $A\mathbf{x} = \mathbf{0}$.

c. Part (b) shows that $\operatorname{Nul} A^T A = \operatorname{Nul} A$. Use this and the rank theorem to show that $\operatorname{rank} A^T A = \operatorname{rank} A$.

Exercise 4. Let

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 0 & 9 & 2 \end{pmatrix}$$

a. Find an eigenvalue/eigenvector pair for A. It is not necessary to compute the characteristic polynomial of A or perform any row reduction.

b. Show that $\det A = 0$. Conclude that 0 is an eigenvalue of A.

c. Use the information gained from parts (a) and (b) to diagonalize A.

Exercise 5. Let A be an $n \times n$ matrix with linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Suppose that B is $n \times n$ and has the same eigenvectors with corresponding eigenvalues $\mu_1, \mu_2, \dots, \mu_n$. Show that $AB = BA$. [*Hint:* Use the given information to diagonalize A and B.]

Exercise 6. Let M be the singular matrix

$$M = \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}.$$

Show that $\det(M + I) = 1 + a + b + c + d$. Using a cofactor expansion is probably not the best idea.

Exercise 7. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$, what value of the scalar x minimizes the quantity $||\mathbf{b} - x\mathbf{a}||$?

Exercise 8. Suppose that the matrices A and B have *the same column space* (they may have different columns, though!).

a. Do A and B necessarily have the same number of pivots?

b. Do A and B necessarily have the same nullspace?

c. If A is invertible, can you conclude that B is invertible as well?

Exercise 9. Suppose that the 3×3 matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -3/4$ and $\lambda_3 = 1/2$. Let $\mathbf{u}_0$ be a vector in $\mathbb{R}^3$ and define the sequence of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots$ by $\mathbf{u}_{k+1} = A\mathbf{u}_k$. Is there a vector $\mathbf{v}$ so that $\mathbf{u}_k \to \mathbf{v}$ as $k \to \infty$? If so, does $\mathbf{v}$ depend on $\mathbf{u}_0$?