Linear Algebra
Fall 2013

Assignment 9.1
Due November 6

Let V be a vector space with basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$, let W be a vector space with basis $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$, and let $T : V \to W$ be a linear transformation. Recall that we defined the matrix of T (relative to $\mathcal{B}$ and $\mathcal{C}$) to be [1]

$$[T]_{\mathcal{B}}^{\mathcal{C}} = ([T(v_1)]_{\mathcal{C}}\ [T(v_2)]_{\mathcal{C}}\ \cdots\ [T(v_n)]_{\mathcal{C}}) \in M_{m\times n},$$

and proved that for any $v \in V$ one has

$$[T(v)]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}}.$$

Exercise 1. Let V, W and T be as above. Prove that T is one-to-one if and only if $[T]$ represents a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e. has a pivot in each column.

Exercise 2. Let V, W and T be as above. Let U be another vector space with basis $\mathcal{D} = \{u_1, u_2, \ldots, u_\ell\}$. Prove that $[S \circ T] = [S][T]$, where on the right-hand side we are using the ordinary matrix product.

Exercise 3. Let H be a subspace of $C^1(\mathbb{R})$ and suppose that H is *closed under differentiation*, i.e. for all $f(x) \in H$, $f'(x) \in H$. Show that for any $a \in \mathbb{R}$ the subspace $V = \{f(x)e^{ax} \mid f(x) \in H\}$ is also closed under differentiation.

Exercise 4. Take $H = \mathbb{P}_3$ and $a = 1$ in the preceding exercise. Find the matrix for the linear transformation $\frac{d}{dx} : V \to V$ relative to the basis $\mathcal{B} = \{e^x, xe^x, x^2e^x, x^3e^x\}$.

Exercise 5. An $n \times n$ matrix N is called *nilpotent* if $N^k = 0$ (the zero matrix) for some $k \geq 1$. Show that if N is nilpotent, then $A = I + N$ is invertible, and give a formula for A^{-1}. [*Suggestion:* Multiply $I + N$ by $I - N + N^2 - N^3 + \cdots + (-1)^{k-1}N^{k-1}$.]

Exercise 6. Use the preceding exercise to invert the matrix $\left[\frac{d}{dx}\right]$ of Exercise 5 *without using row reduction.*

Exercise 7. Use the result of the preceding exercise to compute

$$\int (2x^3 - x^2 + 5)e^x\, dx.$$

[1] We will simply write $[T]$ for this matrix when the bases are clear from context.