Given square matrices $A$ and $B$, we know that in general we do not expect the *commutativity* relationship $AB = BA$ to hold. In this problem we address the question: given $A$, can we determine the $B$ for which $AB = BA$ *is* true, i.e. the matrices $B$ with which $A$ *commutes*?

**Exercise 1.** Let $A$ be a square matrix and let $B = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$, for any choice of real numbers $a_i \in \mathbb{R}$. Show that, in this case, $AB = BA$. We call $B$ a *polynomial in* $A$, and this shows that $A$ always commutes with polynomials in itself.

**Exercise 2.** So, what else might commute with a given square matrix $A$? Given another square matrix $B$, we define the *commutator* of $A$ and $B$ to be $[A, B] = AB - BA$. Show that $AB = BA$ if and only if $[A, B] = 0$.

**Exercise 3.** Fix a square $n \times n$ matrix $A$ and define $C_A : M_{n\times n} \to M_{n\times n}$ by $C_A(X) = [A, X]$. Show that $C_A$ is a linear transformation, and that the kernel of $C_A$ consists of precisely those matrices $X$ that commute with $A$.

**Exercise 4.** Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and define $C_A$ as above. If $\mathcal{B}$ denotes the "standard" basis for $M_{2\times 2}$, find $[C_A]$, the matrix for $C_A$ relative to $\mathcal{B}$.

**Exercise 5.** Find a basis for $\mathrm{Nul}\,[C_A]$. Use $\mathcal{B}$ coordinates to "translate" these back into elements of the kernel of $C_A$.

**Exercise 6.** Show that the matrices you found above (i.e. those that commute with $A$) are all, in fact, polynomials in $A$. This shows that, for this particular matrix, $AB = BA$ if and only if $B$ is a polynomial in $A$.

**Exercise 7.** The preceding result is not true in general. Show that if we take $A = cI$, a scalar multiple of the identity (called a *scalar matrix*), then there are matrices that commute with $A$ that are *not* polynomials in $A$.