LINEAR ALGEBRA
FALL 2013

IN-CLASS EXERCISES
SET 2

Recall that in Calculus II you were introduced to the *Method of Undetermined Coefficients* for solving second order differential equations of the type

$$ay'' + by' + cy = G(x),$$

by making an appropriate "guess" as to the form of $y$, and then solving for its "unknown coefficients." In this set of exercises we will investigate why this procedure always succeeds, as well as provide a "quick" way to implement it.

**Exercise 1.** Prove that if $T(y) = ay'' + by' + cy$ is a linear transformation from the space $C^\infty(\mathbb{R})$ to itself.

**Exercise 2.** Prove that $T$ carries the subspace $H = \{p(x)e^{kx} \mid p(x) \in \mathbb{P}\}$ into itself (here $k$ is an unspecified constant).

**Exercise 3.** Suppose we replace $\mathbb{P}$ above by $\mathbb{P}_4$. Show that $T$ still carries $H$ to itself, and find the matrix $[T]$ of $T$ relative to the basis $\mathcal{B} = \{e^{kx}, xe^{kx}, x^2e^{kx}, x^3e^{kx}, x^4e^{kx}\}$.

**Exercise 4.** Find the determinant of $[T]$. Under what conditions is $T$ invertible?

**Exercise 5.** Show that, when $T$ is invertible, any equation of the form

$$ay'' + by' + c = G(x)$$

has a unique solution $y \in H$, provided $G \in H$ as well.

**Exercise 6.** Choose values of $a, b, c, k$ for which the required invertibility condition holds, and invert $[T]$. Choose several options for $G(x)$ (at least 3), and use $[T]^{-1}$ to help you implement the Method of Undetermined Coefficients, using only matrix multiplication and $\mathcal{B}$-coordinates.

**Exercise 7.** Suppose that $\det[T] = 0$. Do you remember how to modify the Method of Undetermined coefficients in this case? Can you guess how you might be able to modify the approach above to get it to work in that case, too?