Linear Algebra
Fall 2013

In-class exercises
Set 3

Recall that in Calculus II , when you were faced with the need to integrate a rational function (i.e. quotient of polynomials), the algorithm you were taught was to first find the *partial fraction decomposition*, and the integrate term by term. The algebra involved in finding these expansions was frequently quite involved, and in these exercises we will explore a (perhaps) more efficient way to compute them.

Exercise 1. Let $p_1(x), p_2(x), \dots p_n(x) \in \mathbb{P}$ be distinct polynomials, and let $P(x) = p_1(x) \cdot p_2(x) \cdots p_n(x)$. Let $N = \deg P(x)$. Show that $V = \left\{ \frac{q(x)}{P(x)} \,\middle|\, q(x) \in \mathbb{P}_{N-1} \right\}$ is a vector space, with basis $\mathcal{B} = \left\{ \frac{1}{P(x)}, \frac{x}{P(x)}, \dots, \frac{x^{N-1}}{P(x)} \right\}$. [1]

Exercise 2. Finding other bases for V can be complicated when the factors of $P(x)$ are repeated or irreducible quadratic. Rather than deal in complete generality, let's consider one special case. Take $N = 4$, $p_1(x) = x - a$, $p_2 = x - b$, and $p_3 = x^2 + c^2$ (with $a \neq b$, $c \neq 0$). Let $\mathcal{C} = \left\{ \frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x^2+c^2}, \frac{x}{x^2+c^2} \right\}$, and let $W = \operatorname{Span} \mathcal{C}$. Show that W is a subspace of V. Argue, on the basis of dimension, that $V = W$.

Exercise 3. Find the change of basis matrix P from $\mathcal{C}$ to $\mathcal{B}$.

Exercise 4. Invert P to obtain the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Exercise 5. Use the matrix from the preceding exercise to compute the partial fraction decomposition of

$$\frac{2x^2 - 3x + 1}{(x+2)(x+1)(x^2+3)}$$

using only coordinates and matrix multiplication.

Exercise 6. Repeat the preceding exercise for at least two other "randomly" chosen rational functions.

Exercise 7. What happens if $a = b$? What happens if $c = 0$?

[1] You may find it useful to know that the set R of *all* rational functions is a vector space.