Exercise 1. Let $f : A \to B$ and $g : B \to C$. Recall that this means f and g are relations, and in a previous exercise we defined the composition $g \circ f$ of any two relations. We also prove that if f and g are functions, then so is $g \circ f : A \to C$.

a. Show that if f and g are both injective, so is $g \circ f$.

b. Show that if f and g are both surjective, so is $g \circ f$.

c. Show that if f and g are both bijective, so is $g \circ f$.

Exercise 2. Let $f : A \to B$ be a function. Show that if f is invertible (i.e. $f^{-1} : B \to A$ is also a function, then

$$f \circ f^{-1} = \mathrm{Id}_B \text{ and } f^{-1} \circ f = \mathrm{Id}_A,$$

where for any set X, $\mathrm{Id}_X : X \to X$ is the function defined by $\mathrm{Id}_X(x) = x$ for all $x \in X$.

Exercise 3. Let $f : A \to B$ be a function.

a. Show that f is injective if and only if there exists a function $g : B \to A$ so that $g \circ f = \mathrm{Id}_A$. Such a function g is called a *left inverse* of f.

b. Show that f is surjective if and only if there exists a function $h : B \to A$ so that $f \circ h = \mathrm{Id}_B$. Such a function h is called a *right inverse* of f.

c. Use the preceding parts to show that in the case that f is injective, any left inverse of it must be surjective. Furthermore, show that when f is surjective, any right inverse of it must be injective.

c'. [*Optional*] Part **c** can also be derived immediately by proving the following more general statement. If $f : A \to B$ and $g : B \to C$ are functions, and $g \circ f : A \to C$ is a bijection, then f is injective and g is surjective.