INTRODUCTION TO ABSTRACT MATHEMATICS — ASSIGNMENT 11.2

FALL 2013 — DUE NOVEMBER 22

Exercise 1. Recall that in class we proved the following countability criterion.

Proposition 1. *If A is infinite and $f : A \to \mathbb{N}$ is an injection, then A is countable.*

Use this result and exercise 10.2.3 to prove the following complementary criterion.

Proposition. *If A is infinite and $f : \mathbb{N} \to A$ is a surjection, then A is countable.*

Exercise 2.

Here we will complete the technical details of our proof of the countability of $\mathbb{N} \times \mathbb{N}$.

a. Explain why, for each $n \in \mathbb{N}$, there is a unique $d \in \mathbb{N}$ so that

$$\frac{d(d-1)}{2} < n \leq \frac{(d+1)d}{2}.$$

b. Let n and d be as above, and set $k = n - d(d-1)/2$. Show that $k \in \mathbb{N}$.

c. With d, n and k related as above, prove that $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by

$$f(n) = (d - k + 1, k)$$

is a bijection.

d. [*Optional*] Find (and verify!) an explicit formula for the inverse of f.

Exercise 3. Let A be a countable set.

a. Show that $A \times A$ is countable. [*Suggestion:* Use the fact that $\mathbb{N} \times \mathbb{N}$ is countable.]

b. Use induction and part **a** to prove that for all $n \geq 2$, A^n is countable. That is, $|A^n| = |A| = |\mathbb{N}|$.