Introduction to Abstract Mathematics — Assignment 11.3

Fall 2013 — Due November 22

Exercise 1. Let $f_i : A_i \to B_i$ be functions for $i = 1, 2, \dots, n$. Define

$$\begin{aligned}(f_1 \times f_2 \times \cdots \times f_n) : A_1 \times A_2 \times \cdots A_n &\to B_1 \times B_2 \times \cdots \times B_n \\ (a_1, a_2, \dots, a_n) &\mapsto (f(a_1), f(a_2), \dots, f(a_n)).\end{aligned}$$

Prove that $f_1 \times f_2 \times \cdots \times f_n$ is a surjection (resp. injection, bijection) if and only if f_i is a surjection (resp. injection, bijection) for all i.

Exercise 2. Let X be a set and let $f : X \to \mathcal{P}(X)$ be a function. Let

$$Y = \{x \in X \mid x \notin f(x)\}.$$

Prove that for all $x \in X$, $f(x) \neq Y$. Conclude that f *cannot* be surjective.

Exercise 3. We say that a sequence of numbers $\{a_1, a_2, a_3, \dots\}$ is *eventually zero* if there is an $N \in \mathbb{N}$ so that $a_n = 0$ for all $n \geq N$ (N may depend on the sequence). Let $\mathcal{S}_0$ denote the set of all sequences with terms in $\mathbb{N}_0$ that are eventually zero. Prove that $\mathcal{S}_0$ is countable. [*Suggestion:* Consider building a function by using the exponents in the prime factorization of each natural number.]