INTRODUCTION TO ABSTRACT MATHEMATICS — ASSIGNMENT 5.1
FALL 2013 — DUE OCTOBER 4

Exercise 1. In this exercise we will provide another proof that $\sqrt{2}$ is irrational. Assume for the sake of contradiction that $\sqrt{2}$ is rational.

a. Let $p, q \in \mathbb{N}$ with $p/q = \sqrt{2}$. Show that $0 < p - q < q$.

b. With p, q as above, show that

$$\frac{2q - p}{p - q} = \sqrt{2}.$$

c. Use the Well-Ordering Principle to arrive at a contradiction.

Exercise 2. Consider the equation

$$a^2 + b^2 = 3(s^2 + t^2). \tag{1}$$

a. Show that if $a, b, s, t \in \mathbb{N}_0$ satisfy (1), then a and b are both divisible by 3. [*Suggestion:* Use the Lemma of Exercise 2.2.3.]

b. By writing a and b of part **a** as multiples of 3, find a second solution $a', b', s', t' \in \mathbb{N}_0$ to (1).

c. Let $H(x, y, z, w) = x+y+z+w$. Show that if $H(a, b, s, t) > 0$, then $0 < H(a', b', s', t') < H(a, b, s, t)$.

d. Use parts **a**–**c** and the Well-Ordering Principle to give a proof that the only solution to (1) in $\mathbb{N}_0$ is $a = b = s = t = 0$.