Introduction to Abstract Mathematics — Assignment 8.2
Fall 2013 — Due November 1

Exercise 1. Let R be a partial ordering on a set A, and let $B \subseteq A$.

- **a.** Show that $R_B = R \cap (B \times B)$ is a partial ordering on B.
- **b.** If R is a total ordering, is R_B also total?

Exercise 2. If R is a partial ordering on a set A, we say A is *well-ordered* by R if every non-empty $B \subseteq A$ has a least element under R. Show that if A is well-ordered by R, and $C \subseteq A$ is nonempty, then C is well-ordered by R_C (see Exercise 1).

Exercise 3. Given $n \in \mathbb{N}$, define $\nu_2(n)$ to be the exponent of the largest power of 2 that divides n. So, for example, $\nu_2(4) = 2$, $\nu_2(15) = 0$, $\nu_2(56) = 3$. Define the relation $\triangleleft$ on $\mathbb{N}$ by

$$\triangleleft = \{(a, b) \mid (\nu_2(a) < \nu_2(b)) \vee (\nu_2(a) = \nu_2(b) \text{ and } a \leq b)\}.$$

Prove that $\triangleleft$ is a total ordering on $\mathbb{N}$. [1]

Exercise 4. Let $\triangleleft$ denote the ordering on $\mathbb{N}$ defined in Exercise 3. Let E denote the set of all even natural numbers. Under $\triangleleft$, determine the set $L(E)$ of lower bounds of E. Find the greatest element of $L(E)$, i.e. the greatest lower bound of E.

[1] $\triangleleft$ is known as the Sharkovskii ordering, and is important in the study of discrete dynamical systems.