Exercise 1. Consider the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

at a point $P = (p, q)$ in the first quadrant.

a. Find the x and y intercepts of the tangent line at P.

First we construct the tangent line. Since we already have the coordinates of P, we only need to find the derivative at P. We implicitly differentiate (1):

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{xb^2}{ya^2} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_P = -\frac{pb^2}{qa^2}.$$

Therefore the tangent line has point-slope equation

$$y - q = -\frac{pb^2}{qa^2}(x - p).$$

To obtain the x-intercept we set $y = 0$ and solve for x:

$$\begin{aligned} -q &= -\frac{pb^2}{qa^2}(x - p) \\ \frac{q^2a^2}{pb^2} &= x - p \\ x &= p + \frac{q^2a^2}{pb^2} = \frac{p^2b^2 + q^2a^2}{pb^2} \\ &= \frac{a^2b^2\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)}{pb^2} \\ &= \boxed{\frac{a^2}{p}}, \end{aligned}$$

where we have used the fact that since P lies on the ellipse, its coordinates must satisfy (1). Either by setting $x = 0$ in the equation for the tangent line and solving for y (as above), or arguing by symmetry, we likewise find that the y-intercept is

$$\boxed{\frac{b^2}{q}}.$$

b. Find the minimum length of the portion tangent line cut off by the coordinate axes.

To simplify things, we parametrize the points P on the ellipse by the unit circle, setting $p = a\cos\theta$, $q = b\sin\theta$ with $0 < \theta < \pi/2$. Using the intercepts found in part **a**, the

Pythagorean Theorem tells us that the length, L, of the portion of the tangent line in the first quadrant satisfies

$$L^2 = \left(\frac{a^2}{p}\right)^2 + \left(\frac{b^2}{q}\right)^2 = \frac{a^4}{a^2\cos^2\theta} + \frac{b^4}{b^2\sin^2\theta} = a^2\sec^2\theta + b^2\csc^2\theta.$$

Differentiating we obtain

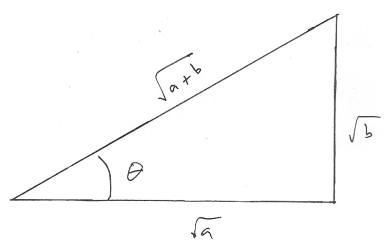
$$2L\frac{dL}{d\theta} = 2a^2\sec^2\theta\tan\theta - 2b^2\csc^2\theta\cot\theta = \frac{2a^2\sin\theta}{\cos^3\theta} - \frac{2b^2\cos\theta}{\sin^3\theta} = \frac{2a^2\sin^4\theta - 2b^2\cos^4\theta}{\cos^3\theta\sin^3\theta}$$

so that

$$\frac{dL}{d\theta} = \frac{(a\sin^2\theta - b\cos^2\theta)(a\sin^2\theta + b\cos^2\theta)}{L\cos^3\theta\sin^3\theta} = \frac{a\cos^2\theta(\tan^2\theta - b/a)(a\sin^2\theta + b\cos^2\theta)}{L\cos^3\theta\sin^3\theta}.$$

Since $a, b > 0$, $\cos\theta$, $\sin\theta$ are positive on $(0, \pi/2)$, and $\tan^2\theta$ is increasing there as well, we find that there is a single critical point when $\tan^2\theta = b/a$, that $\frac{dL}{d\theta}$ is negative to its left, and is positive to its right. Hence the global minimum value of L occurs when $\tan^2\theta = b/a$.

To evaluate L, we represent the relationship $\tan^2\theta = b/a$ graphically with the right triangle shown below.



From it we deduce that

$$\sec^2\theta = \frac{a+b}{a} \quad \text{and} \quad \csc^2\theta = \frac{a+b}{b}$$

and hence

$$L^2 = a^2\sec^2\theta + b^2\csc^2\theta = a(a+b) + b(a+b) = (a+b)^2 \quad \Rightarrow \quad \boxed{L = a+b}.$$

c. Minimize the area of the triangle formed by the tangent line and the coordinate axes.

By part **a**, the (right) triangle formed by the tangent line and the coordinate axes has width a^2/p and height b^2/q, so that its area is

$$A = \frac{a^2b^2}{2pq} = \frac{a^2b^2\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)}{2pq} = \frac{p^2b^2 + q^2a^2}{2pq} = \frac{1}{2}\left(a^2\frac{q}{p} + b^2\frac{p}{q}\right).$$

The quantity $m = q/p$ is the slope of the line segment between P and the origin. Any (non-vertical) line through the origin is totally determined by its slope, and if that slope is positive it will strike the ellipse at a unique point point P in the first quadrant.

In other words, we may replace P by $m = q/p$ throughout the problem and will still find the same maximum value of A. Making this substitution we find that we need to optimize

$$A = \frac{1}{2}\left(a^2 m + \frac{b^2}{m}\right)$$

on the open interval $0 < m < \infty$. We have

$$\frac{dA}{dm} = \frac{1}{2}\left(a^2 - \frac{b^2}{m^2}\right) = \frac{a^2m^2 - b^2}{2m^2} = \frac{(am-b)(am+b)}{2m^2}.$$

Since $a, b, m > 0$, $m = b/a$ is the only critical point in the domain of A, and we see that $\frac{dA}{dm}$ is negative to its left, and positive to its right. Hence the absolute minimum of A occurs when $m = b/a$, and this minimum value is

$$A\left(\frac{b}{a}\right) = \frac{1}{2}\left(a^2\frac{b}{a} + \frac{b^2}{b/a}\right) = \frac{1}{2}(ab + ab) = \boxed{ab}.$$