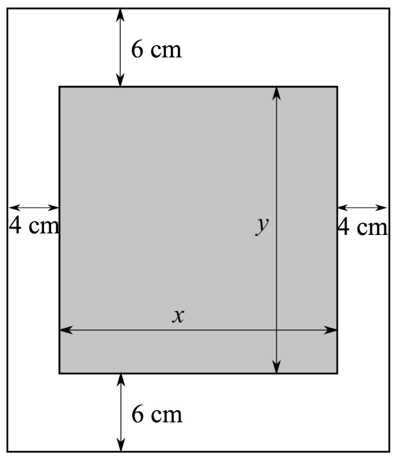
Calculus I
Fall 2017

Applied Optimization Examples

Example 1. The top and bottom margins of a poster are each 6 cm and the side margins are 4 cm. If the area of the printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



Let x and y denote the dimensions of the printed portion of the poster, so that

$$xy = 384 \quad \Rightarrow \quad y = \frac{384}{x}$$

and the overall area is given by

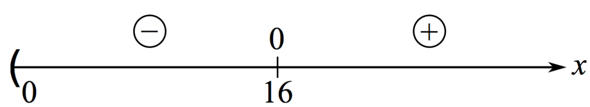
$$A = (x+8)(y+12) = (x+8)\left(\frac{384}{x} + 12\right) = 384 + \frac{3072}{x} + 12x + 96 = 12x + \frac{3072}{x} + 480,$$

with domain $0 < x < \infty$.

The critical numbers of A are given by

$$0 = A' = 12 - \frac{3072}{x^2} \iff x^2 = 256 \iff x = \pm 16.$$

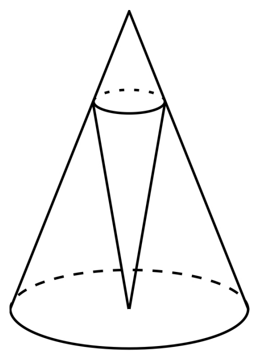
We may discard $x = -16$ since it isn't in our domain, leaving us with the single critical number $x = 16$. Looking at the sign of A' we have



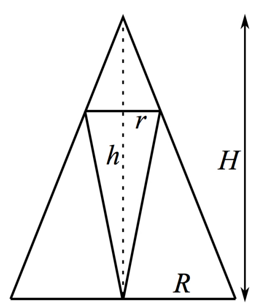
Since $A' > 0$ to the right of $x = 16$ and negative to the left, A must have an absolute minimum value at $x = 16$. This corresponds to $y = 24$ for overall dimensions

$$\boxed{24 \text{ cm} \times 36 \text{ cm}}.$$

Example 2. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.



Taking a vertical cross-section through the vertices of both cones yields the following diagram.



The top-most triangle is similar to the large triangle, so that

$$\frac{H-h}{r} = \frac{H}{R} \quad \Rightarrow \quad r = \frac{R}{H}(H-h).$$

Hence the volume of the inscribed cone is

$$V = \pi r^2 h = \frac{\pi R^2}{H^2}(H-h)^2 h.$$

Here H and R are fixed positive constants and $0 \le h \le H$. So we may use the closed interval method.

To locate the critical numbers we set

$$\begin{aligned}
0 &= \frac{dV}{dh} \\
&= \frac{\pi R^2}{H^2}\left(-2(H-h)h + (H-h)^2\right) \\
&= \frac{\pi R^2}{H^2}(H-h)(-2h+H-h) \\
&= \frac{\pi R^2}{H^2}(H-h)(H-3h) \\
\Rightarrow &\quad h = H/3, H.
\end{aligned}$$

Since $V(0) = V(H) = 0$, the maximum value must occur when $h = H/3$:

$$V(H/3) = \frac{4\pi R^2 H}{27},$$

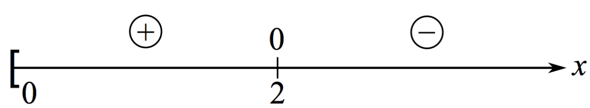
which is just less than half the volume of the larger cone.

Example 3. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

Since the slope of the tangent line is just the derivative, we need to maximize $y' = 120x^2 - 15x^4 = f(x)$ on the entire real line $(-\infty, \infty)$. However, since f is even, we need only consider $[0, \infty)$. Now according to Fermat's Theorem, the absolute maximum (if it exists) occurs somewhere $f'(x) = 0$, i.e. where

$$f'(x) = 240x - 60x^3 = 60x(4 - x^2) = 0 \iff x = 0, \pm 2 \;\Rightarrow x = 0, 2.$$

It is easy to see that the sign of f' is given by



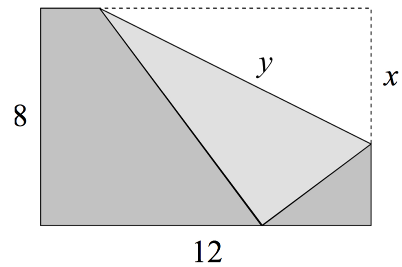
Since $f'(x)$ is positive for $x < 2$ and negative for $x > 2$, $f(2)$ must be the absolute maximum value of f on $[0, \infty)$.

By the symmetry of even functions, the same maximum value occurs at $x = -2$. This and $x = 2$ give the points

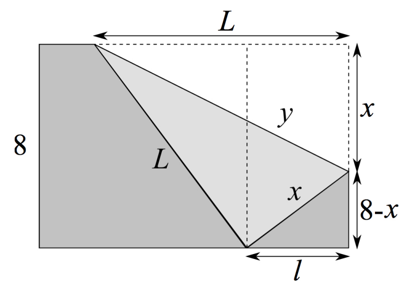
$$\boxed{(2, 225) \text{ and } (-2, -223)}$$

on the original curve.

Example 4. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



Introduce the lengths ℓ and L as shown below.



There are now three right triangles in the diagram. The Pythagorean Theorem applied to the bottom-right triangle yields

$$x^2 = \ell^2 + (8-x)^2 = \ell^2 + 64 - 16x + x^2 \quad \Rightarrow \quad \ell^2 = 16(x-4).$$

When applied to the top-left triangle, then Pythagorean Theorem gives

$$L^2 = (L-\ell)^2 + 8^2 = L^2 - 2L\ell + \ell^2 + 64 \quad \Rightarrow \quad L = \frac{\ell}{2} + \frac{32}{\ell} = \frac{\ell^2 + 64}{2\ell}.$$

Finally, applying the Pythagorean Theorem to the "folded" triangle, and using the results just obtained, we have

$$y^2 = x^2 + L^2 = x^2 + \frac{\left(\ell^2 + 64\right)^2}{4\ell^2} = x^2 + \frac{(16x)^2}{64(x-4)} = x^2 + \frac{4x^2}{x-4}.$$

In order for the top-right corner of the paper to fold down to touch the bottom edge and form a triangle, we must have $x > 4$, and certainly x can't exceed the height of the paper, so $x < 8$. Hence the domain of y is

$$4 < x < 8,$$

an open interval.

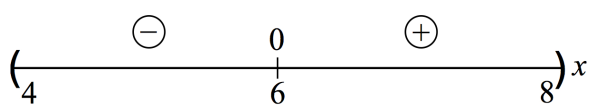
Now to find the critical number of y. We could solve for y above but instead we simply implicitly differentiate, obtaining

$$\begin{aligned}
2y\frac{dy}{dx} &= 2x + \frac{(x-4)\cdot 8x - 4x^2}{(x-4)^2} \\
&= 2x + \frac{4x^2 - 32x}{(x-4)^2} \\
&= 2x + \frac{4x(x-8)}{(x-4)^2} \\
&= 2x\left(1 + \frac{2(x-8)}{(x-4)^2}\right) \\
&= 2x\left(\frac{(x-4)^2 + 2(x-8)}{(x-4)^2}\right) \\
&= 2x\left(\frac{x^2 - 6x}{(x-4)^2}\right) \\
&= 2x^2\left(\frac{x-6}{(x-4)^2}\right).
\end{aligned}$$

Hence, since $y > 0$

$$\frac{dy}{dx} = \frac{x^2}{y}\left(\frac{x-6}{(x-4)^2}\right).$$

This is only zero in our domain when $x = 6$, and because every factor other than $x - 6$ is always positive, dy/dx is positive when $x > 6$ and negative when $x < 6$. That is, we have the following sign diagram for dy/dx.



Hence, since there is only one critical point, the First Derivative Test guarantees that we have an absolute minimum when $\boxed{x = 6}$. The corresponding y value is $6\sqrt{3}$.