Algebra II
Fall 2017

Assignment 1.2
Due August 30

Exercise 1. Consider the set

$$\mathbb{H} = \{a + bi + cj + dk \,|\, a, b, c, d \in \mathbb{R}\}$$

of all formal linear combinations of the (linearly independent) symbols 1, i, j and k.

- **a.** If we declare that $i^2 = j^2 = k^2 = ijk = -1$, show that $ij = k$, $jk = i$, $ki = j$ and that i, j, k *anti-commute*, i.e. satisfy $xy = -yx$.
- **b.** Using component-wise addition and polynomial multiplication (subject to the relationships of part **a**), $\mathbb{H}$ becomes a ring known as the *(Hamiltonian) quaternions* (you don't need to prove this). Given $x = a + bi + cj + dk \in \mathbb{H}$, we define its *conjugate* to be $\overline{x} = a - bi - cj - dk$. Show that

$$x\overline{x} = \overline{x}x = a^2 + b^2 + c^2 + d^2.$$

- **c.** Use part **b** to show that $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$.

Exercise 2. Let R be a commutative ring with unity, $a, b \in R$.

- **a.** Show that if a is a unit and $b^2 = 0$, then $a + b$ is also a unit.
- **b.** Show that if a and b are both *nilpotent* ($a^m = b^n = 0$ for some $m, n \in \mathbb{N}$), then $a + b$ is also nilpotent.

Exercise 3. Find all linear and quadratic units in $\mathbb{Z}_4[x]$.

Exercise 4. Let R be a ring with unity and suppose $a \in R$ satisfies $a^2 = 1$. Prove that $S = aRa = \{ara \,|\, r \in R\}$ is a subring of R with unity.

Exercise 5. Prove Theorem 12.1.

Exercise 6. Let R be a ring. Prove that each of the following hypotheses imply R is commutative.

- **a.** $a^2 - b^2 = (a - b)(a + b)$ for all $a, b \in R$.

b. $(R, +)$ is cyclic.

c. $a^2 = a$ for all $a \in R$.