Exercise 1. Let R be a commutative ring with prime characteristic p. Prove that the Frobenius map $a \mapsto a^p$ is an endomorphism of R.

Exercise 2. Let

$$R = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

a. Show that R is a subring of $\mathrm{M}_2(\mathbb{R})$.

b. Show that the map $\phi : \mathbb{C} \to R$ given by

$$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is an isomorphism.

Exercise 3. Let R and S be rings, $\phi : R \to S$ a homomorphism.

a. Show that $\phi(R)$ is a subring of S.

b. Show that if R and S have unity and ϕ is surjective, then $\phi(1) = 1$.

c. Show that if I is an ideal of S, then

$$J = \phi^{-1}(I) = \{r \in R \,|\, \phi(r) \in I\}$$

is an ideal of R.

Exercise 4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an automorphism. Let $x, y \in \mathbb{R}$.

a. Show that $\phi(r) = r$ for all $r \in \mathbb{Q}$.

b. Show that if $x > 0$, then $\phi(x) > 0$. [*Suggestion:* Every positive element of $\mathbb{R}$ is a square.]

c. Use part **b** to show that if $x > y$, then $\phi(x) > \phi(y)$.

d. Suppose that $x < \phi(x)$ and choose $r \in \mathbb{Q}$ so that $x < r < \phi(x)$. Use parts **a** and **c** to arrive at a contradiction. A similar contradiction may be reached if $\phi(x) < x$. What does this tell us about ϕ?