Algebra II
Fall 2017

Assignment 5.2
Due September 27

Exercise 1. Let R be a commutative ring and S a multiplicative subset. Let $q/s, q'/s', r/t, r'/t' \in S^{-1}R$. If $q/s = q'/s'$ and $r/t = r'/t'$, prove that

$$\frac{q}{s} + \frac{r}{t} = \frac{q'}{s'} + \frac{r'}{t'} \quad \text{and} \quad \frac{q}{s} \cdot \frac{r}{t} = \frac{q'}{s'} \cdot \frac{r'}{t'}$$

using the definitions of addition and multiplication of fraction equivalence classes.[1]

Exercise 2. Let R be a commutative ring, $S \subseteq R \setminus \{0\}$ a multiplicative set. Suppose that S contains no zero divisors. Let $s \in S$ and define $\varphi_s : R \to S^{-1}R$ by $r \mapsto rs/s$.

a. Prove that $\varphi_s = \varphi_t$ for any $t \in S$.

b. Prove that φ_s is a monomorphism. Thus $S^{-1}R$ contains (a copy of) R.

c. Prove that $\varphi_s(t)$ is a unit in $S^{-1}R$ for all $t \in S$. So $S^{-1}R$ is an extension of R in which the elements of S become invertible.

Exercise 3. Let D be a subdomain of a field F. Let $Q = \{rs^{-1} \in F \,|\, r, s \in D, s \neq 0\}$. Prove that Q is the smallest subfield of F containing D, i.e. Q is contained in every subfield of F that contains D.

Exercise 4. Let D be a domain and $S \subseteq D \setminus \{0\}$ be a multiplicative set.

a. If $I \subseteq S^{-1}D$ is an ideal, consider the "numerators of I":

$$J = \{d \in D \,|\, \exists s \in S \text{ such that } d/s \in I\}.$$

Show that J is an ideal in D.

b. Continuing the notation of part **a**, show that if J is principal (generated by a single element), then so is I.

c. A domain in which every ideal is principal is called a *principal ideal domain* (PID). The prototypical example of a PID is $\mathbb{Z}$. Use parts **a** and **b** to show that if D is a PID, then so is $S^{-1}D$.

[1] Have no fear: this isn't nearly as tedious as I remember it being almost 20 years ago.