Exercise 1. Let R be a commutative ring and $S \subseteq R$ a multiplicative set. Let $U \subset R^\times$ and set $T = U \cup S$. Show that the map $\phi : S^{-1}R \to T^{-1}R$ given by $\phi(r/s) = r/s$ is a well-defined isomorphism of rings. In other words, forming quotients with units has no effect.

Exercise 2. Let R be a commutative ring, $f \in R[x]$. We define the (formal) derivative of f as follows. If $f \in R$ we set $f' = 0$. Otherwise, write $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $n \in \mathbb{N}$ and $a_n \neq 0$, and set

$$f' = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

a. Show that the derivative is R-linear, i.e. if $r, s \in R$ and $f, g \in R[x]$, then

$$(rf + sg)' = rf' + sg'.$$

b. Show that the derivative obeys the *Leibniz rule*, i.e. for all $f, g \in R[x]$

$$(fg)' = fg' + f'g.$$

[*Suggestion:* Fix f and induct on $\deg g$.]

Exercise 3. Let F be a field, $a \in F$ and $f \in F[x]$. Show that a is a multiple root of f (i.e. has multiplicity at least 2) if and only if $f(a) = f'(a) = 0$.