Exercise 1. Let R be a ring and M an R-module. Let $A, B \subset M$ be linearly independent and disjoint. Show that $S = A \cup B$ is linearly independent if and only if $\text{Span}(A) \cap \text{Span}(B) = \{0\}$.

Exercise 2. Let R be a ring and M an R-module. Suppose that $\mathcal{C} = \{X_\alpha\}$ is a *chain* of linearly independent subsets of M, i.e. if $X_\alpha, X_\beta \in \mathcal{C}$, then $X_\alpha \subseteq X_\beta$ or $X_\beta \subseteq X_\alpha$.

a. Show that if $n \geq 2$ and $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_n} \in \mathcal{C}$, then there is a k so that $X_{\alpha_i} \subseteq X_{\alpha_k}$ for all i. [*Suggestion:* Induct on n.]

a. Show that $\bigcup_{X \in \mathcal{C}} X$ is linearly independent.

Exercise 3. Let V be a vector space over F. Show that if S is a maximal linearly independent subset of V, then S spans V, and is consequently a basis for V. Where does your proof break down if F is merely a commutative ring with unity? [*Suggestion:* Let $v \in V$. If $v \notin S$, then $S \cup \{v\}$ must be linearly dependent, by maximality of S. Use this to express v in terms of S.]